MAX-PLANCK-INSTITUT FÜR KERNPHYSIK

An Introduction to Particle Acceleration

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Learning outcomes

- Non-thermal particle acceleration essentials
- How to generate non-thermal power-laws
- How to determine maximum energies

Assumed Knowledge

- Basics of electromagnetism, relativity, hydrodynamics
- Basics of vector algebra, calculus concepts, including Taylor series

Lecture Overview

- Non-thermal emission from astrophysical systems
- Particle acceleration essentials
- Enrico Fermi’s great insight
- Diffusive Shock Acceleration
- A quick digression into plasma physics

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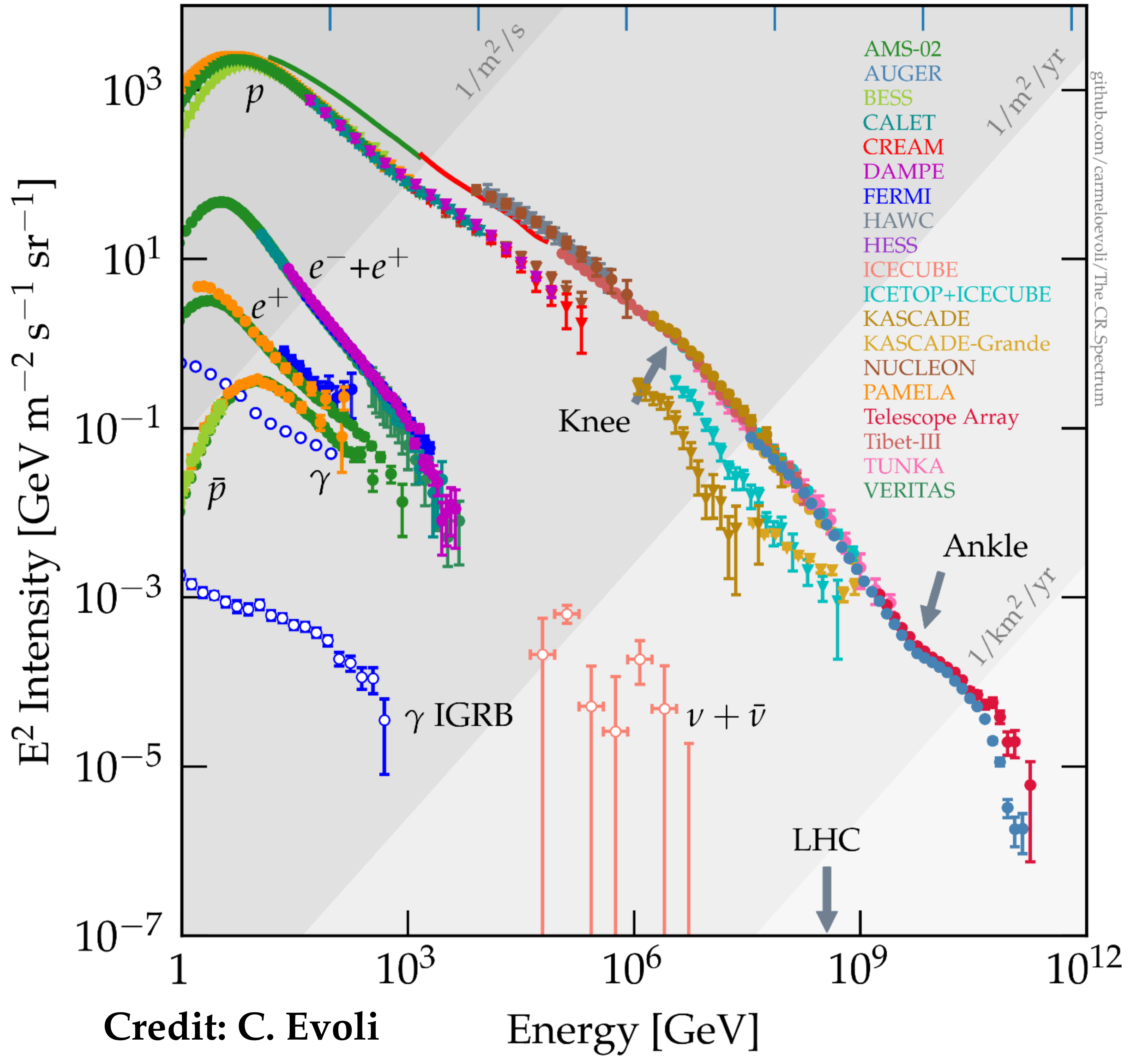

The spectrum of Cosmic rays measured at Earth

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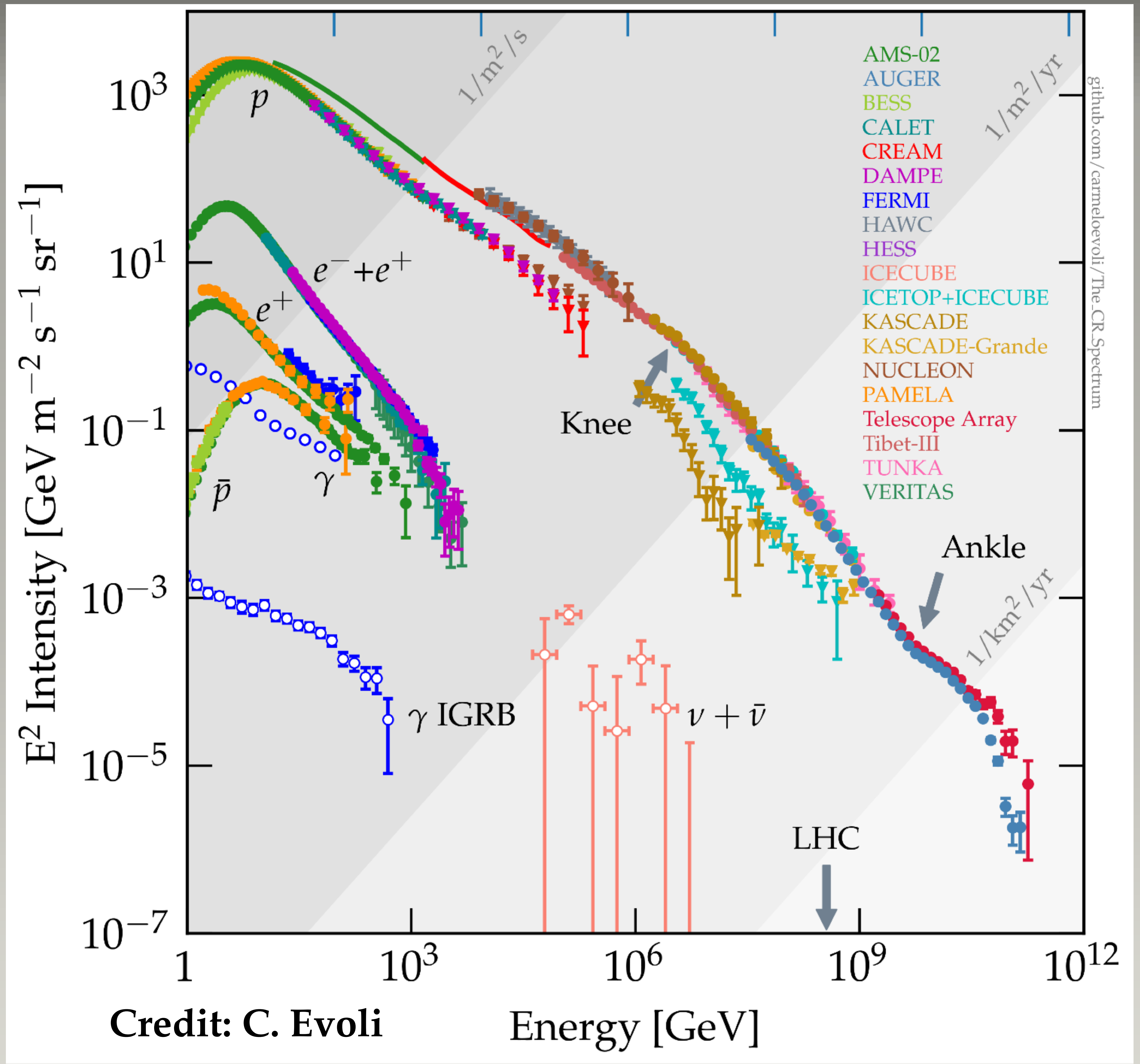

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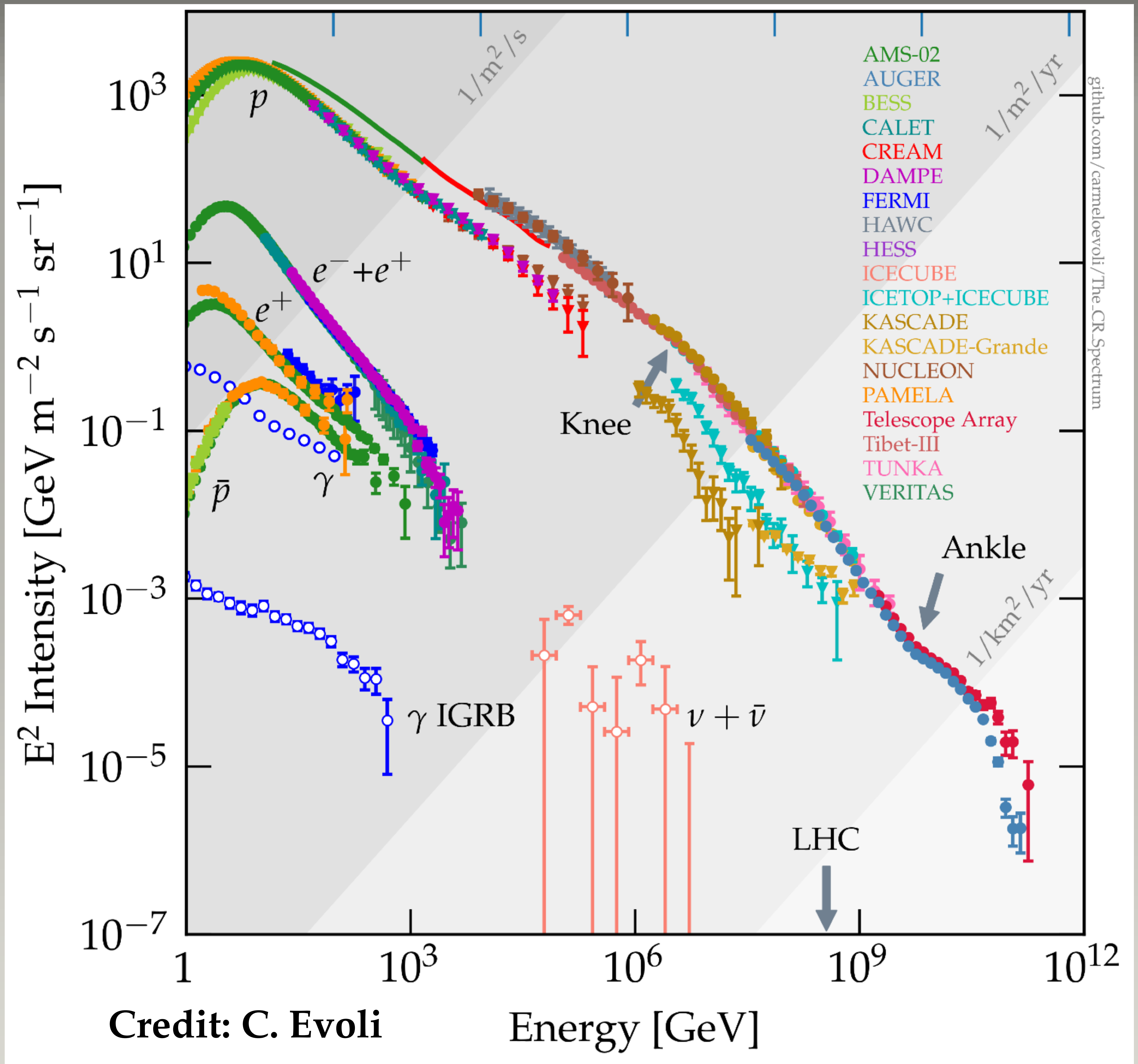

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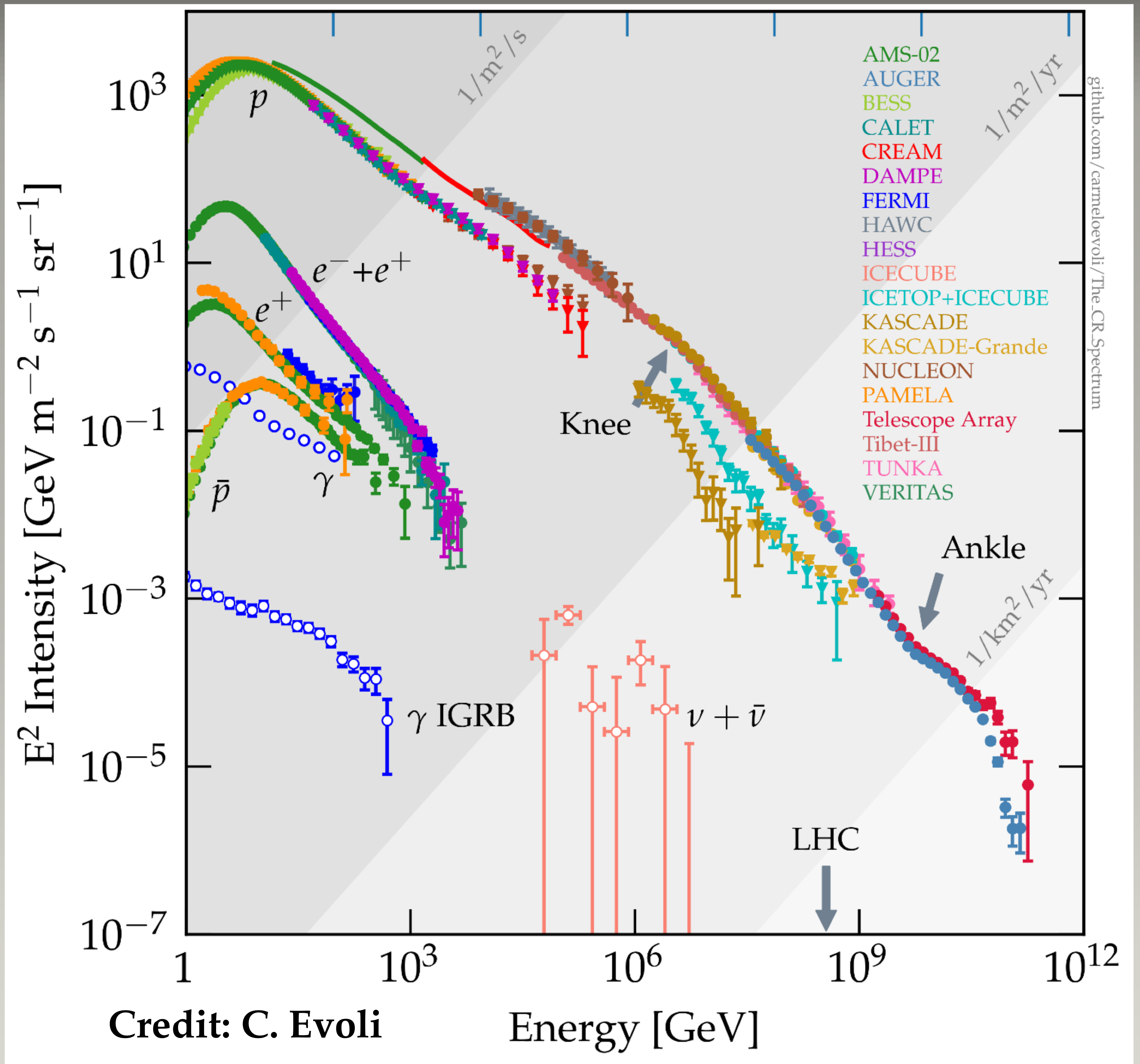

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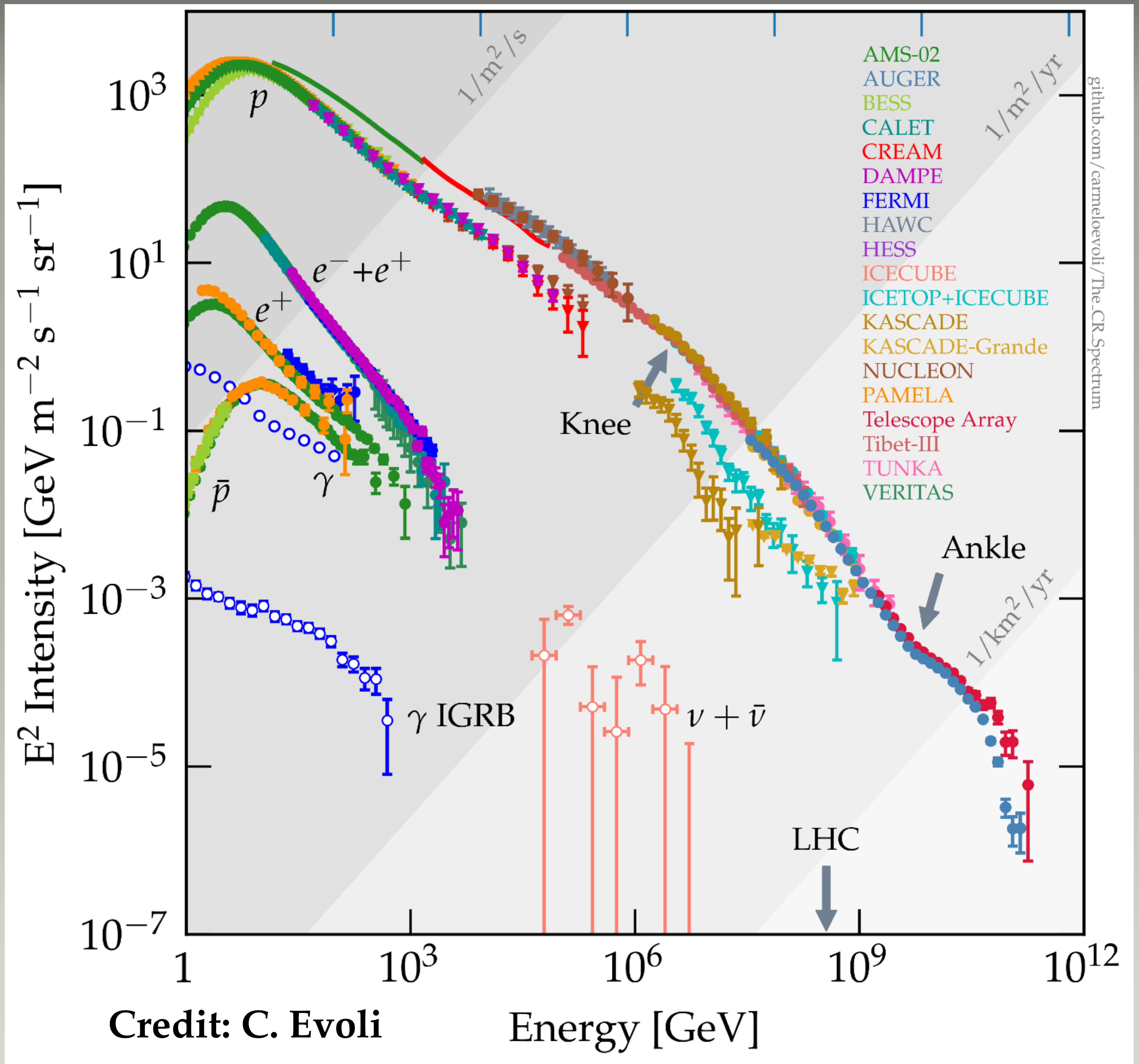

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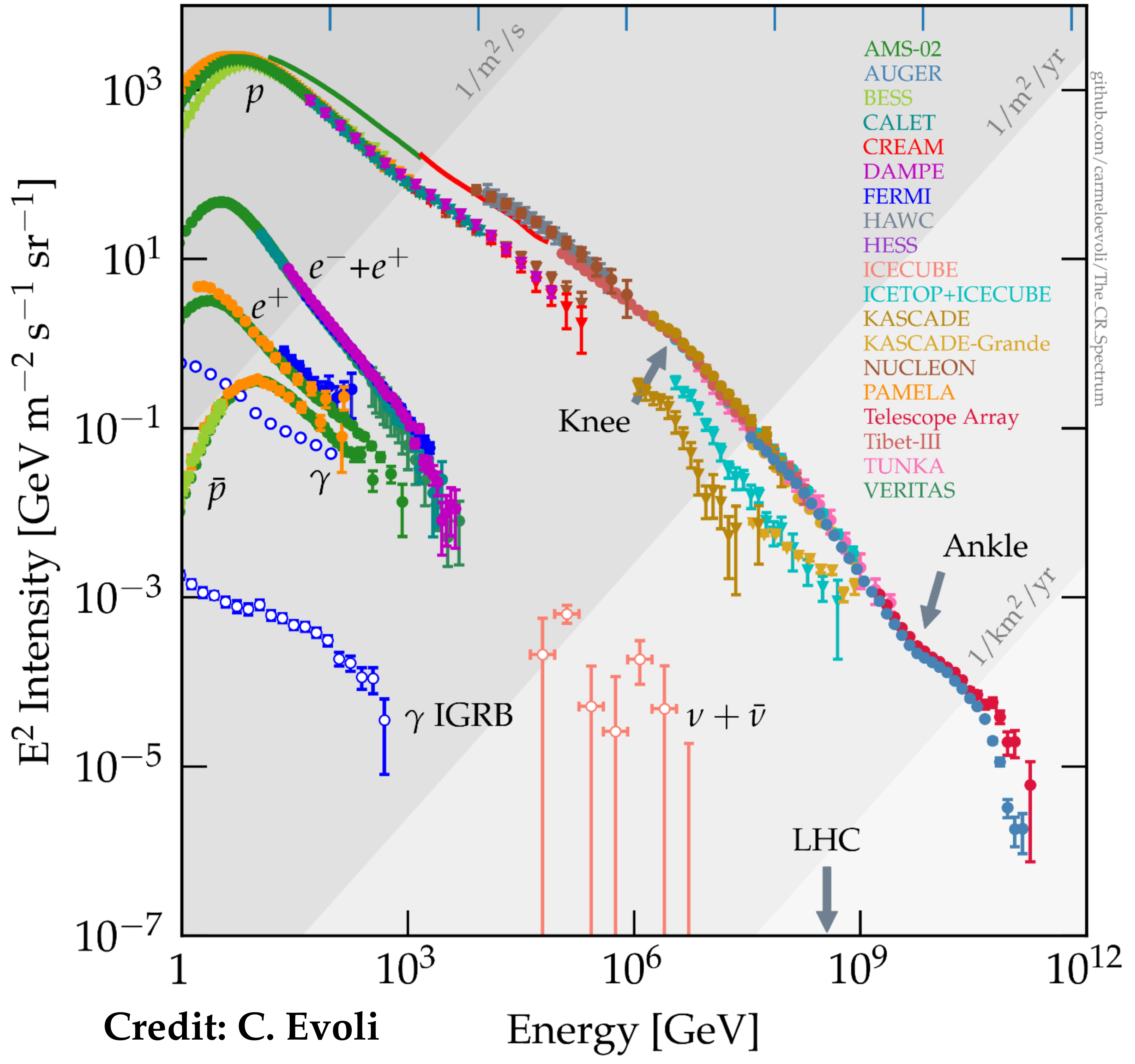

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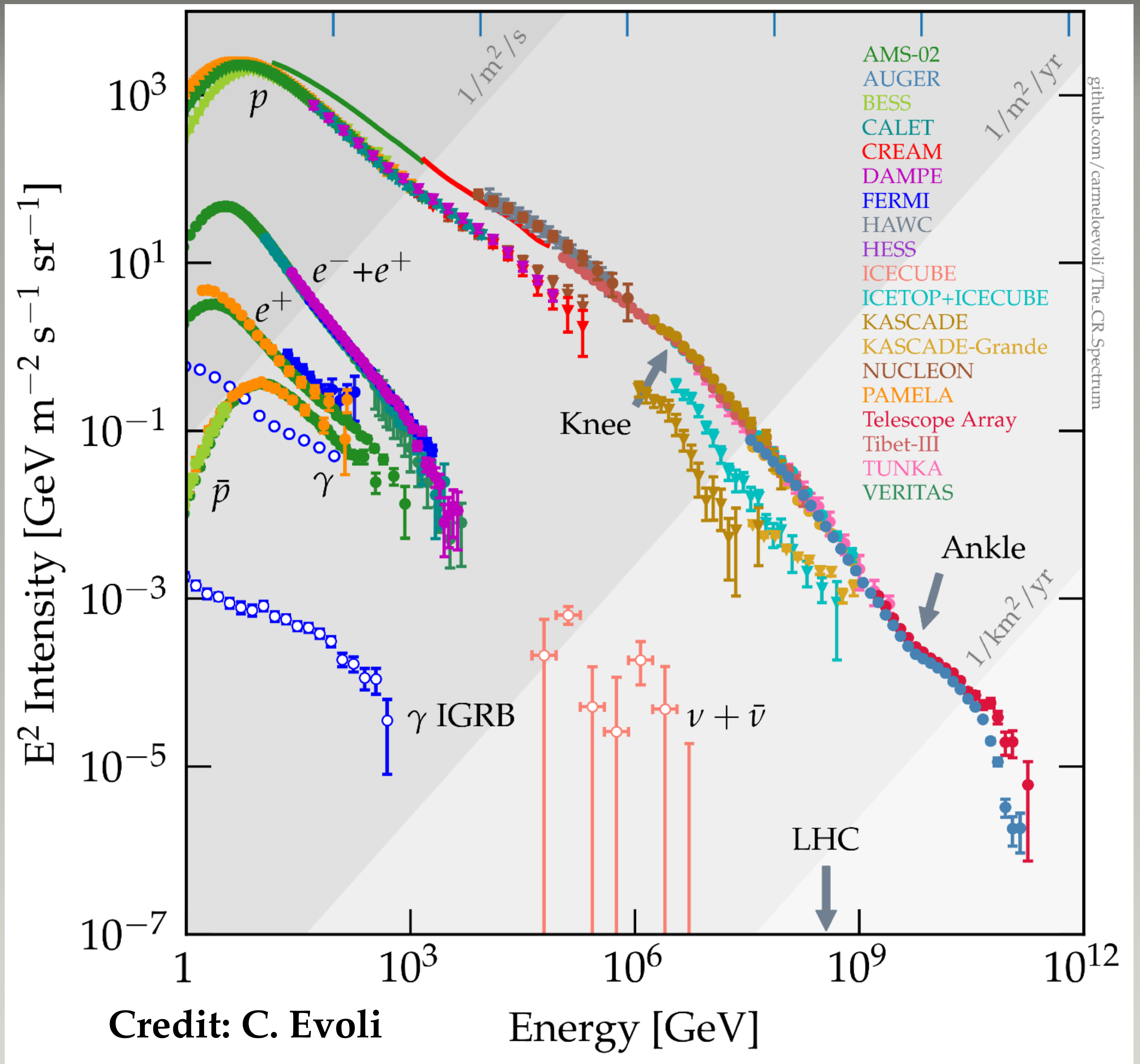

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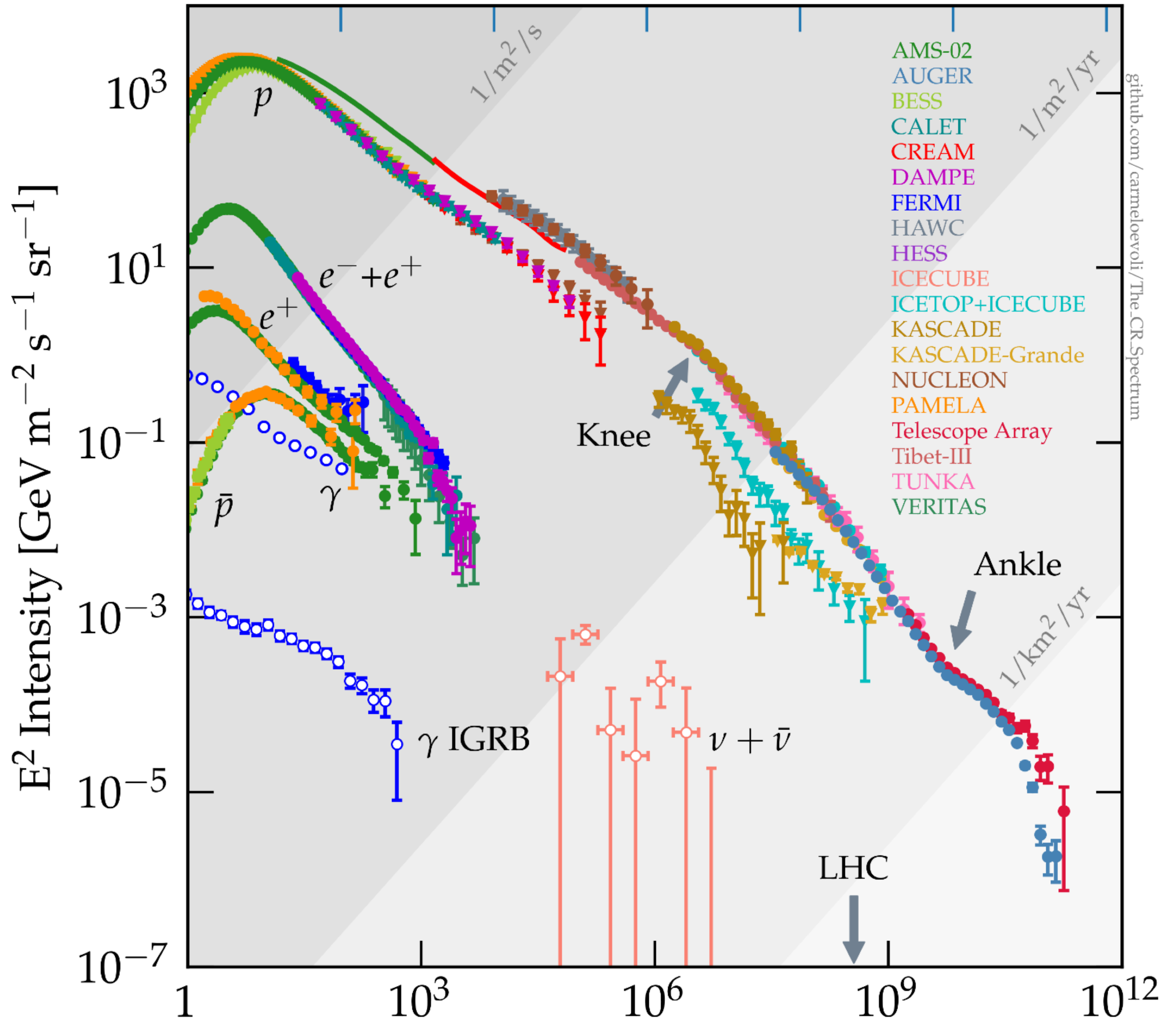
- **CRs above the ankle have gyro radius > galactic disk height -> extragalactic in origin**

Searching for Cosmic Ray sources



Credit: C. Evoli

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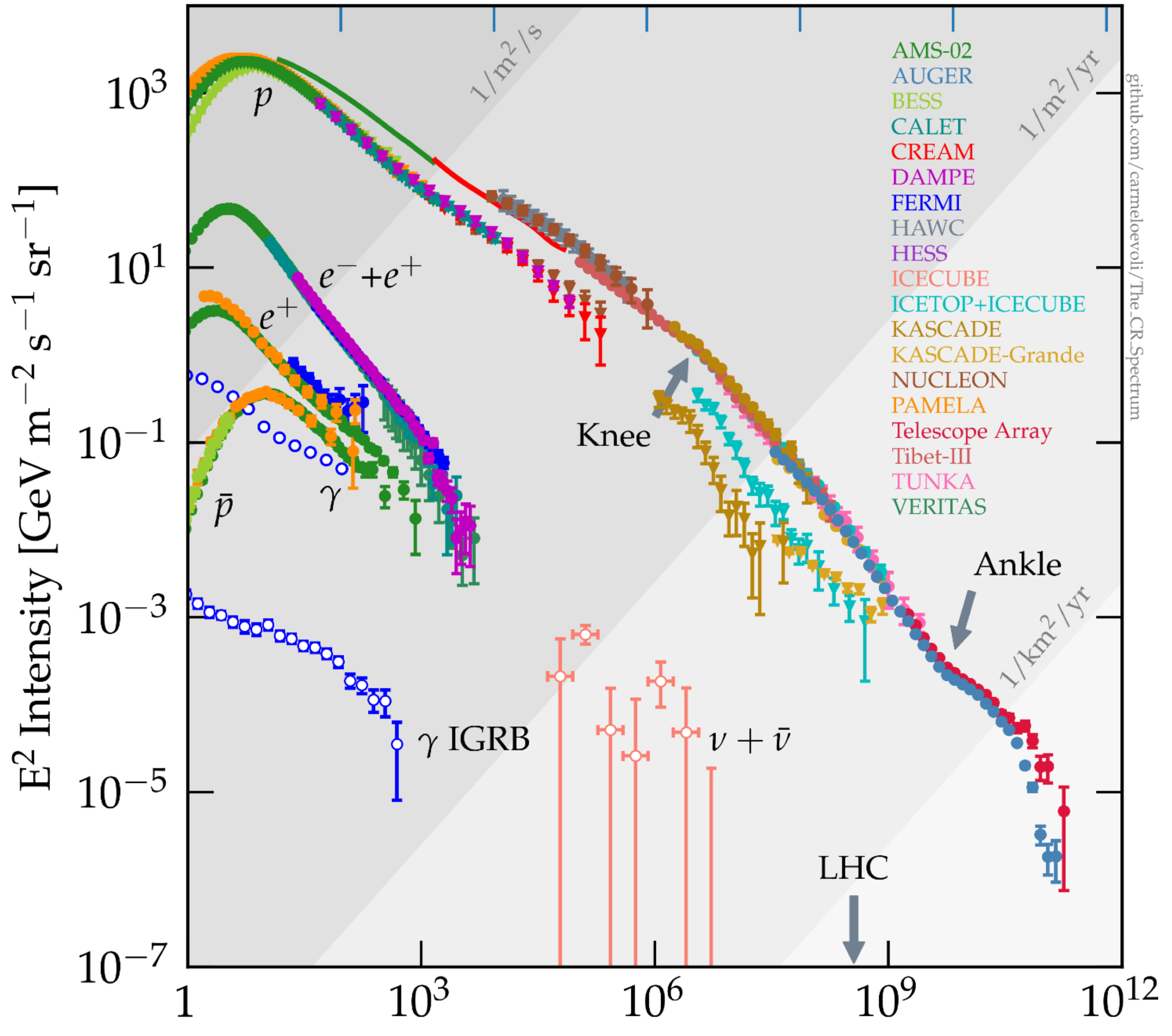

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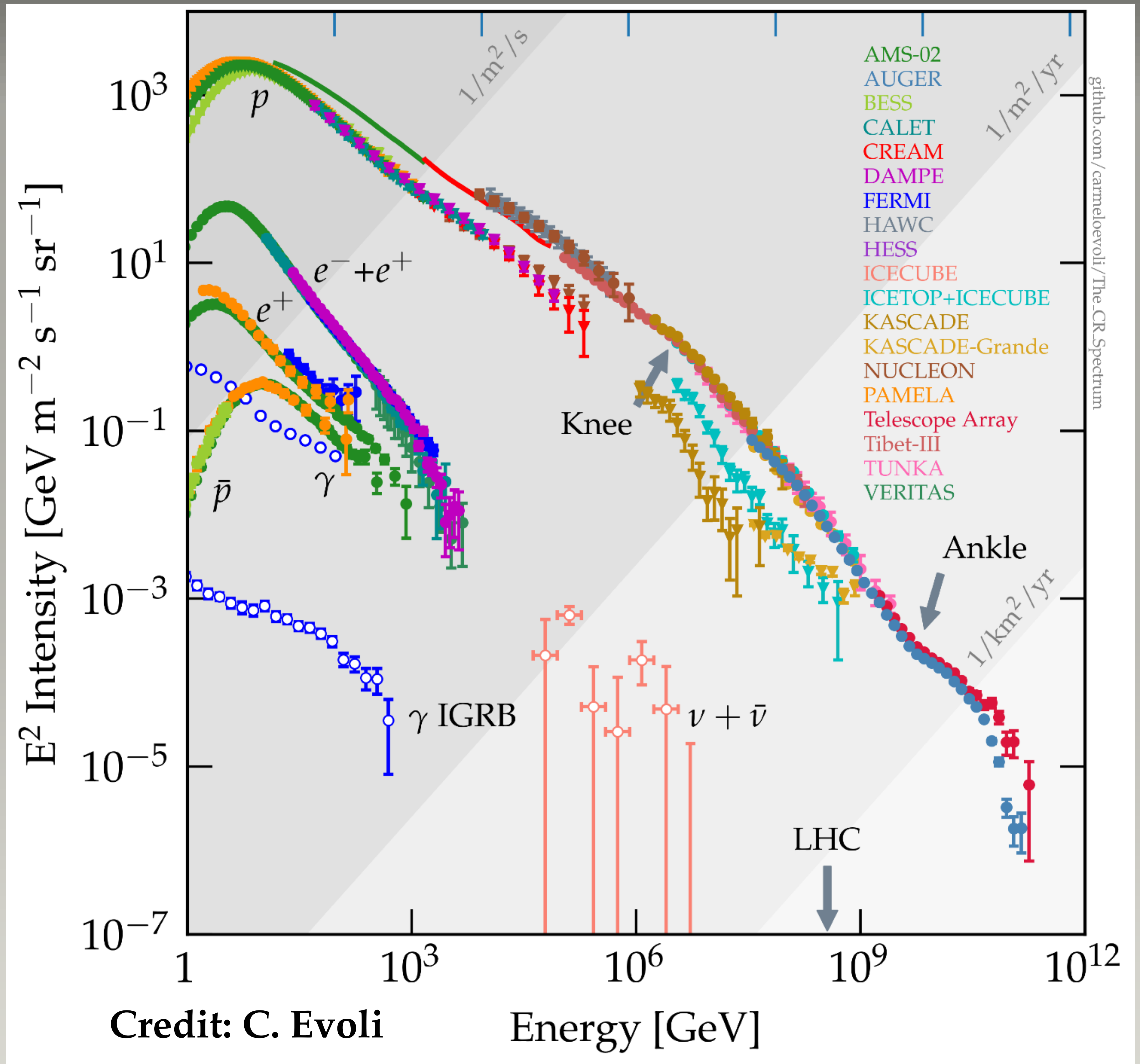

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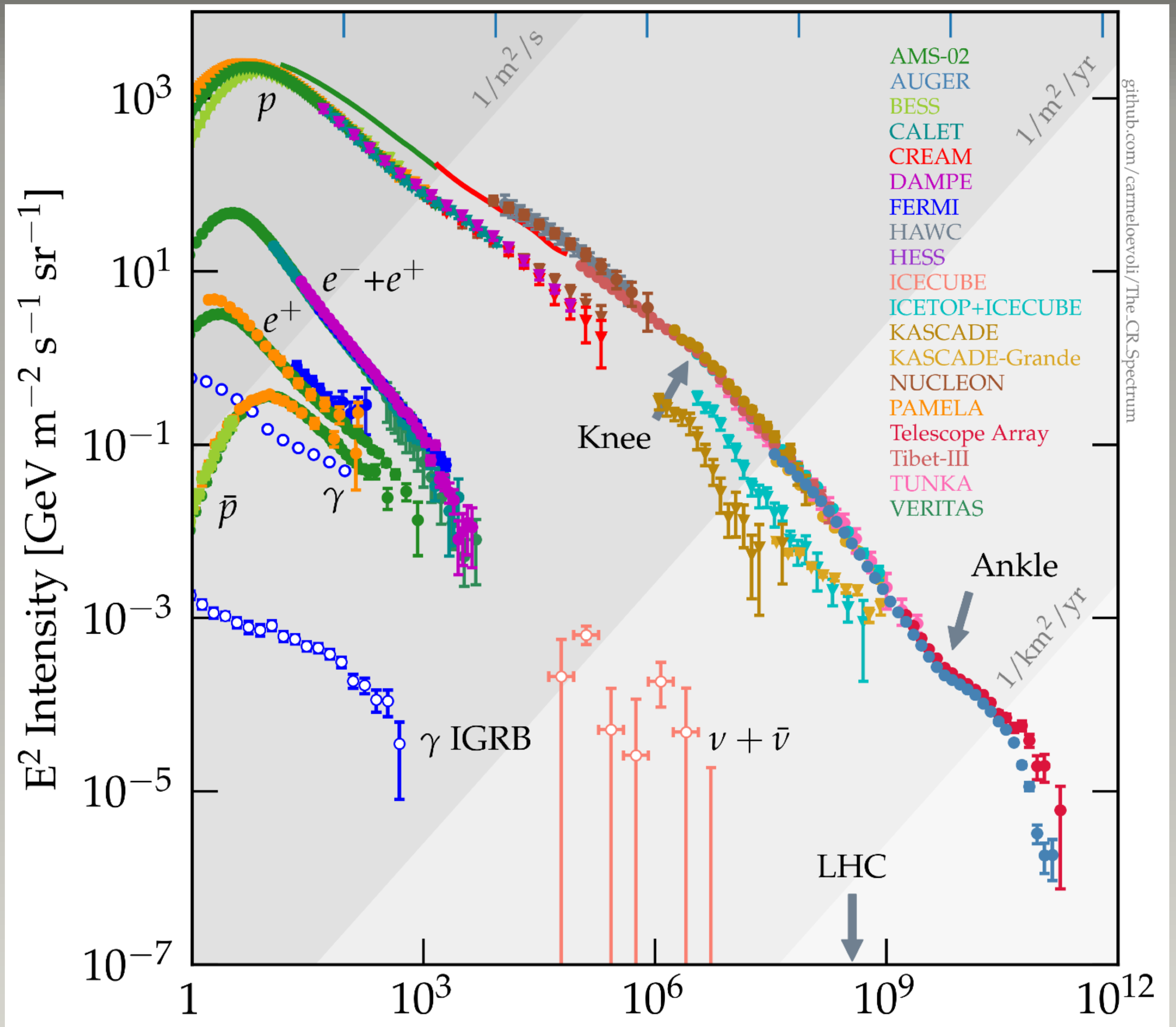

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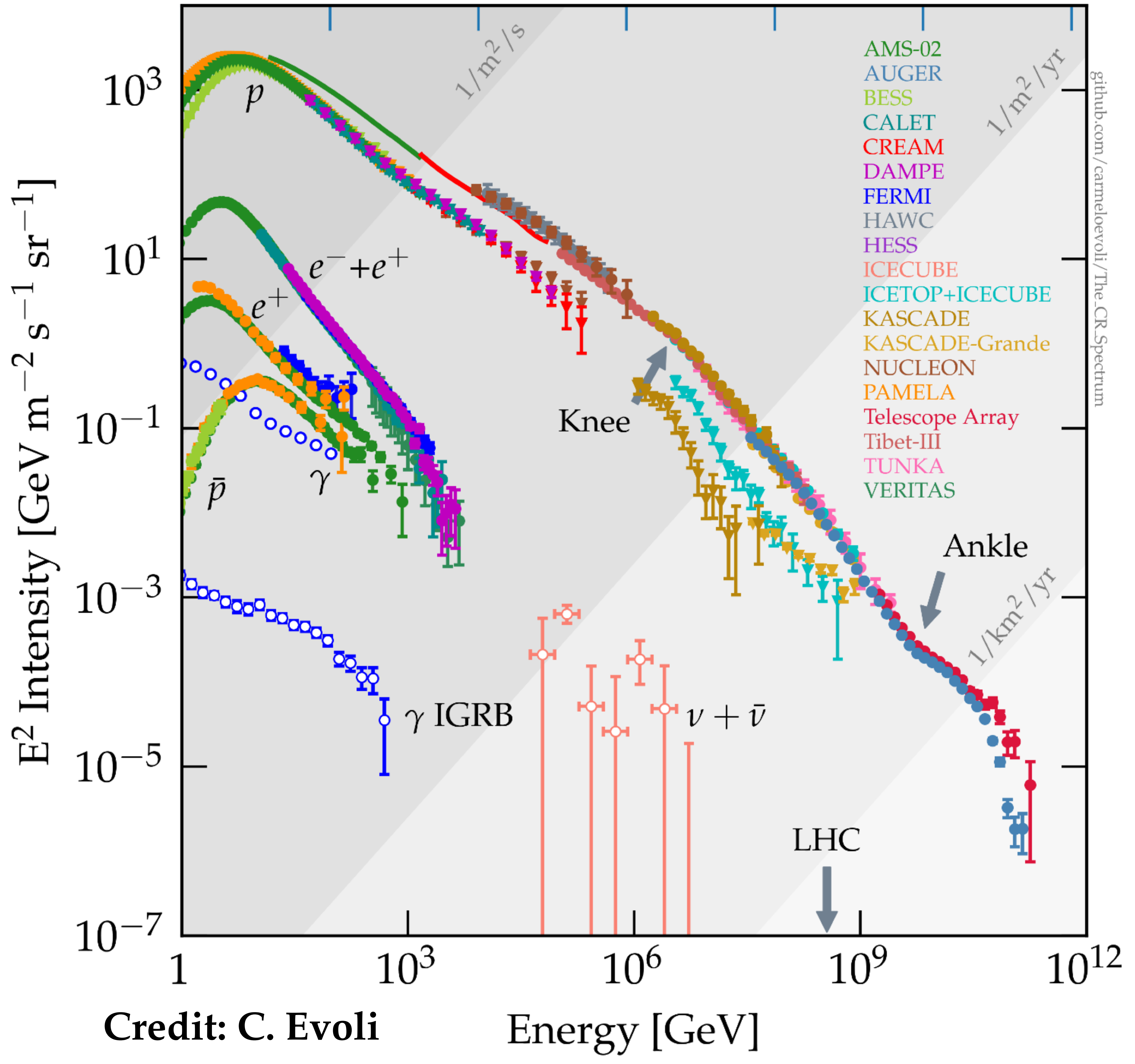

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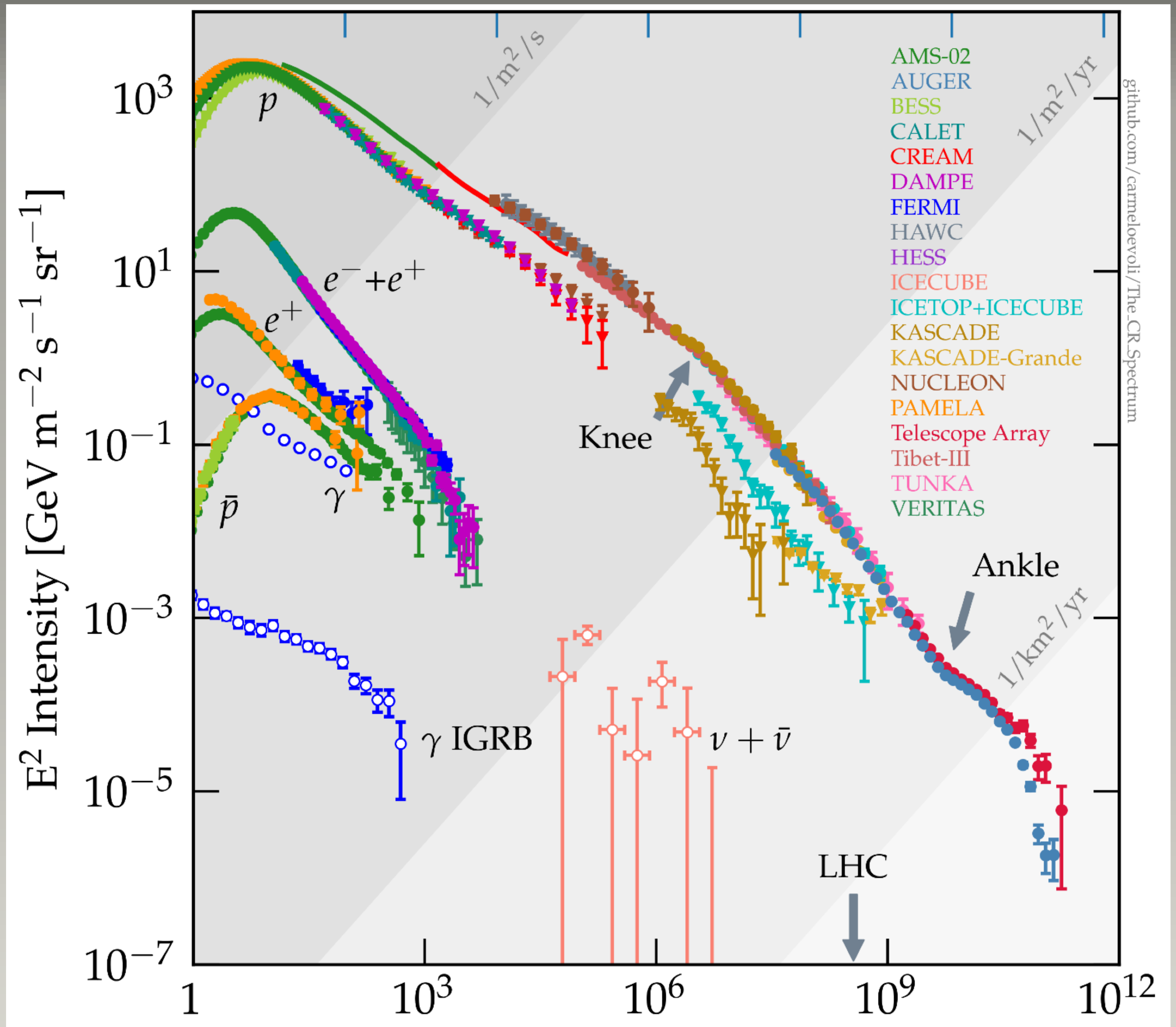

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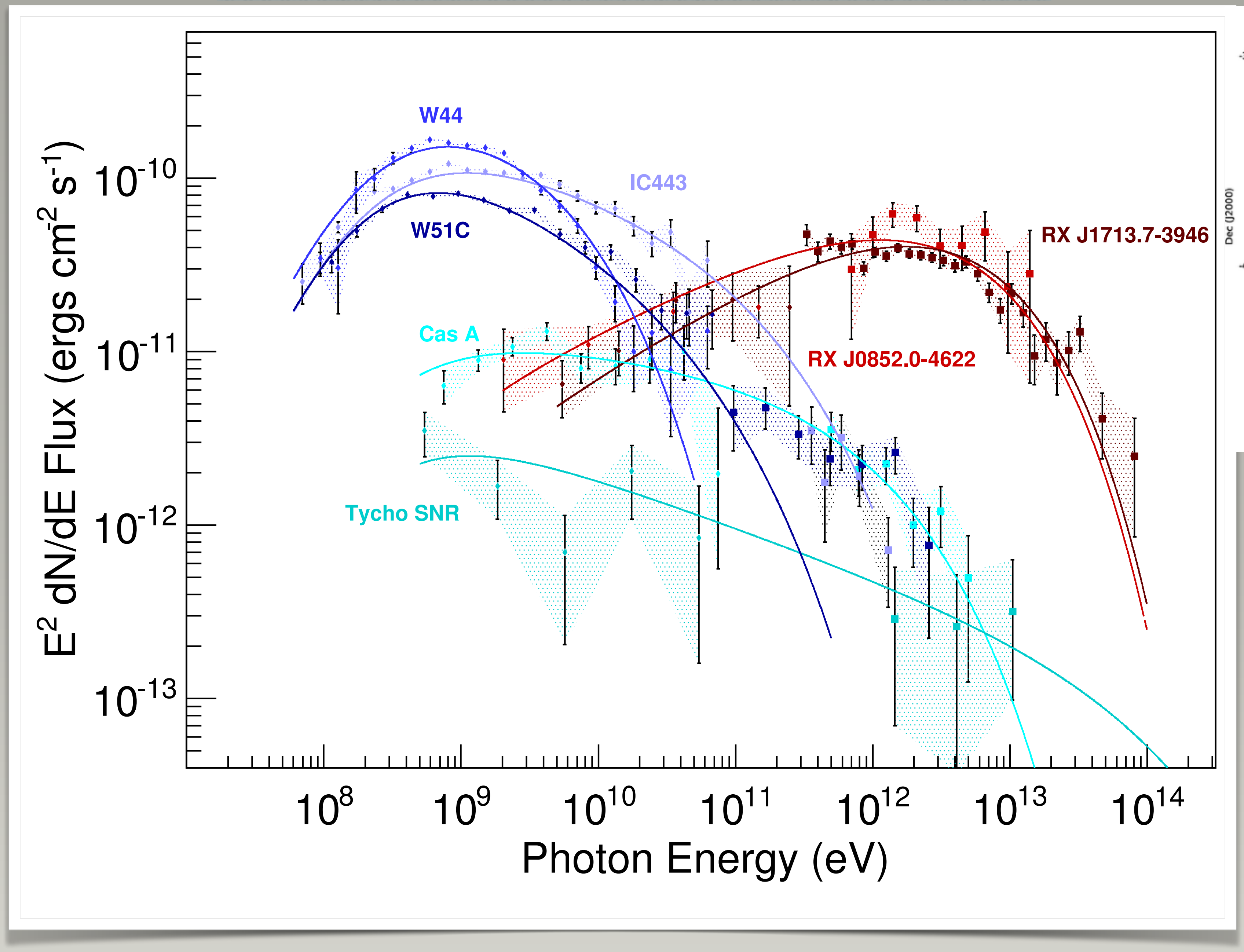
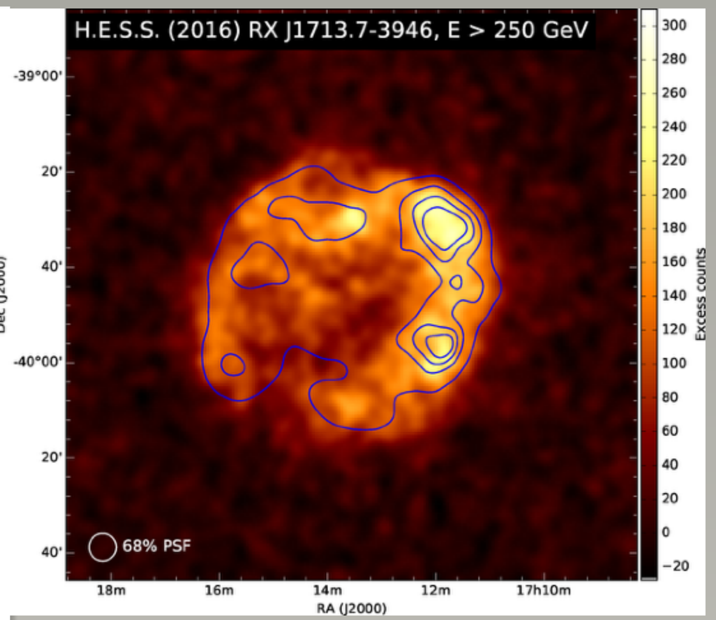
- **Process is typically inefficient and competing with more efficient processes like Bremsstrahlung or Inverse Compton from energetic electrons**

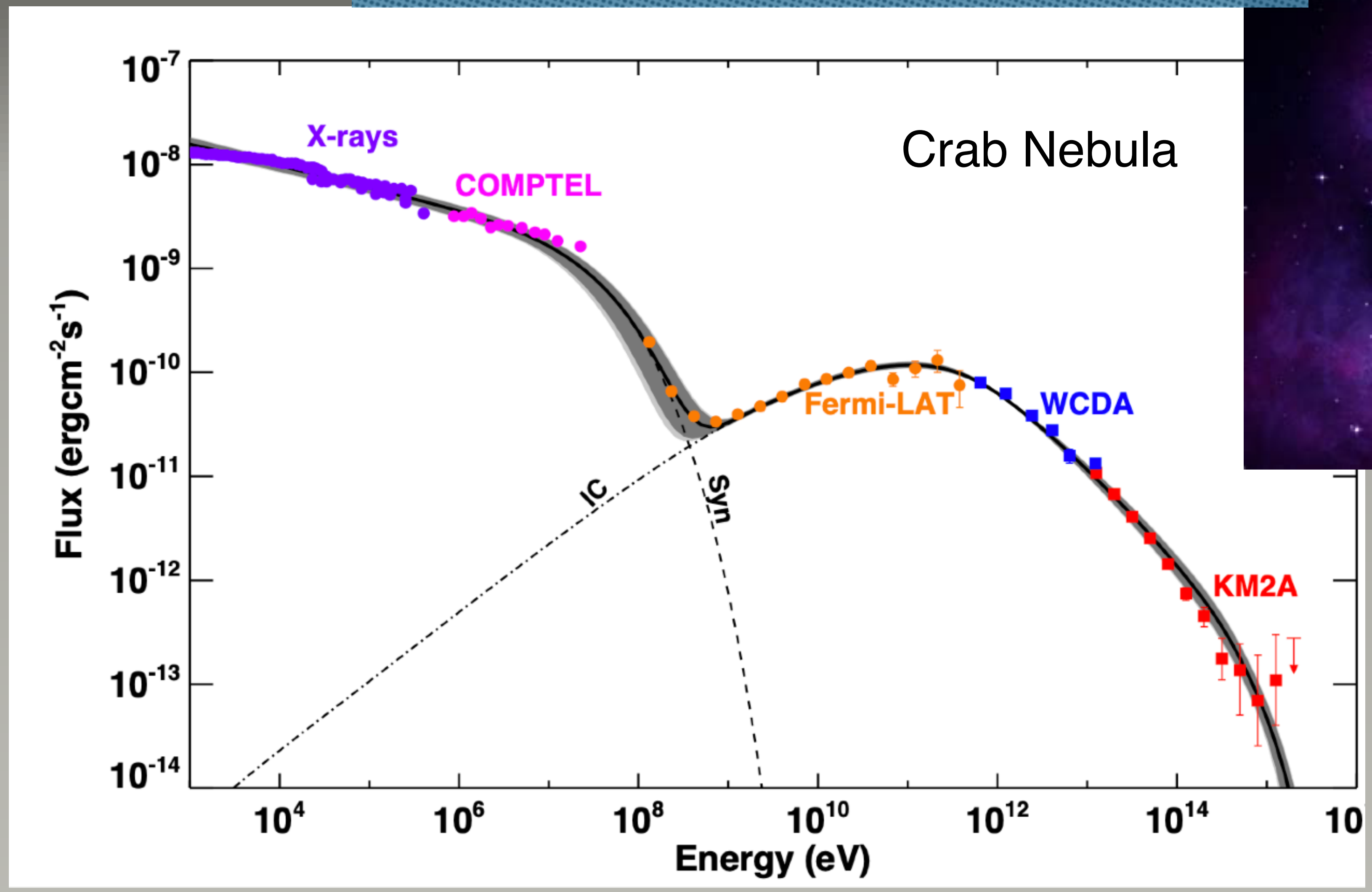
Searching for Cosmic Ray sources

Supernova Remnants

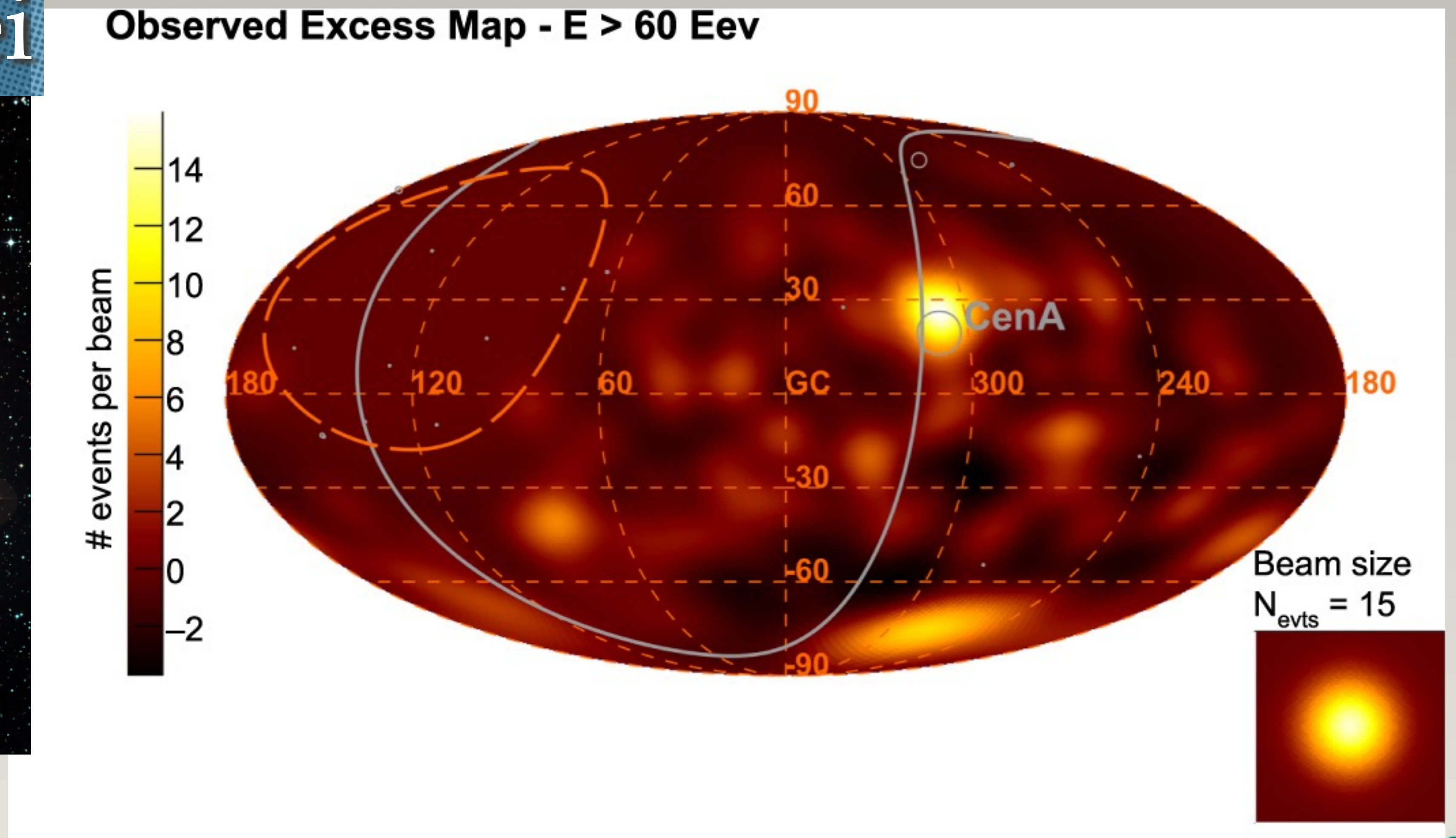




Pulsar Wind Nebulae



Active Galactic Nuclei



Credit: S. Funk, LHAASO, HESS. NASA, Auger Coll.

Searching for Cosmic Ray sources

To produce high energy photons/neutrinos, we need high energy particles

How does Nature do this?

Lecture Overview

- Non-thermal emission from astrophysical systems
- **Particle acceleration essentials**
- Enrico Fermi’s great insight
- Diffusive Shock Acceleration
- A quick digression into plasma physics
- Relativistic outflows

Astrophysical vs Terrestrial Gas

	Air	ISM
Density	$10^{20}\ \mathrm{cm}^{-3}$	$0.1 - 1\ \mathrm{cm}^{-3}$
Temperature	$300\ \mathrm{K}$	$\sim 10^4\ \mathrm{K}\ \ (\sim 1\ \mathrm{eV})$
τ_{eq}	$\sim$ ns	$\sim$ year

The air you are breathing, to a very good approximation, satisfies Maxwell-Boltzmann statistics.

Astrophysical plasmas are "*collisionless*" on timescales << τ_{eq}

When system is heated/perturbed, non-thermal populations can result.

But why/how are some particles so shamefully greedy???

What accelerates particles?

Consider a flat space-time. The equations of motion for a particle of charge q are:

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \qquad \frac{d\mathbf{x}}{dt} = \mathbf{v}$$

where $\mathbf{p} = \gamma m \mathbf{v}$ **and** $\gamma = \dfrac{1}{\sqrt{1 - |\mathbf{v}|^2/c^2}}$

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Only electric field does work on a particle. Magnetic fields alone can not accelerate particles.

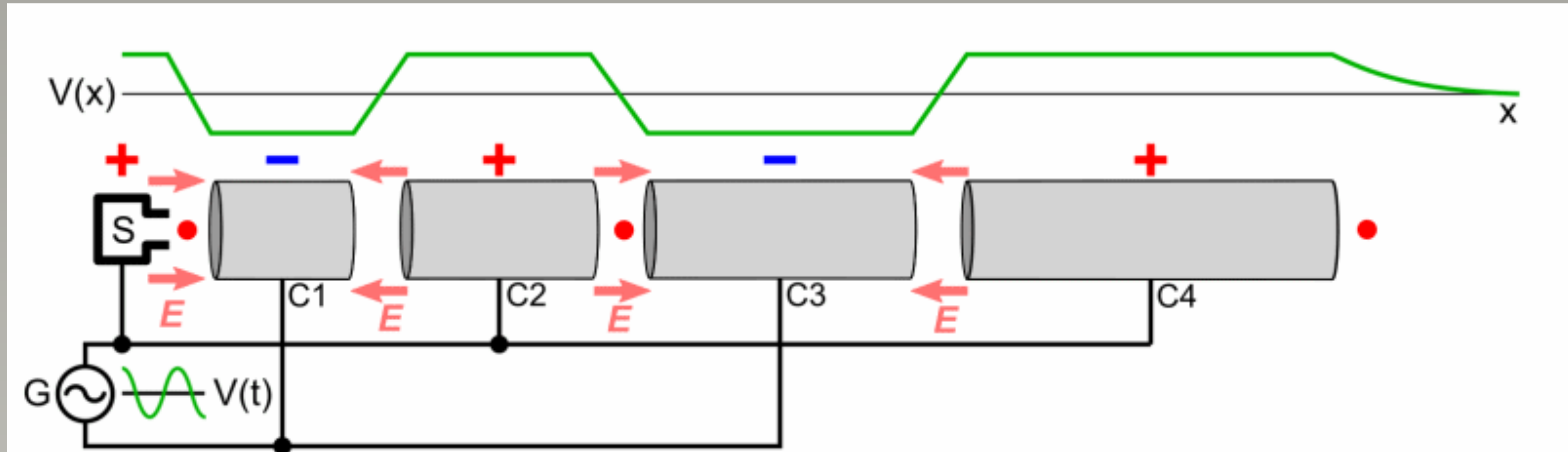
$$\varepsilon_{\max} = q\int \mathbf{E} \cdot \mathbf{v}\,dt = q\int \mathbf{E} \cdot d\mathbf{s}$$

What electric fields are expected in astrophysical systems?

Regular vs Stochastic Acceleration

In terrestrial laboratories/experiments, we exploit charge “gaps”

e.g. a radio frequency (RF) linear accelerator



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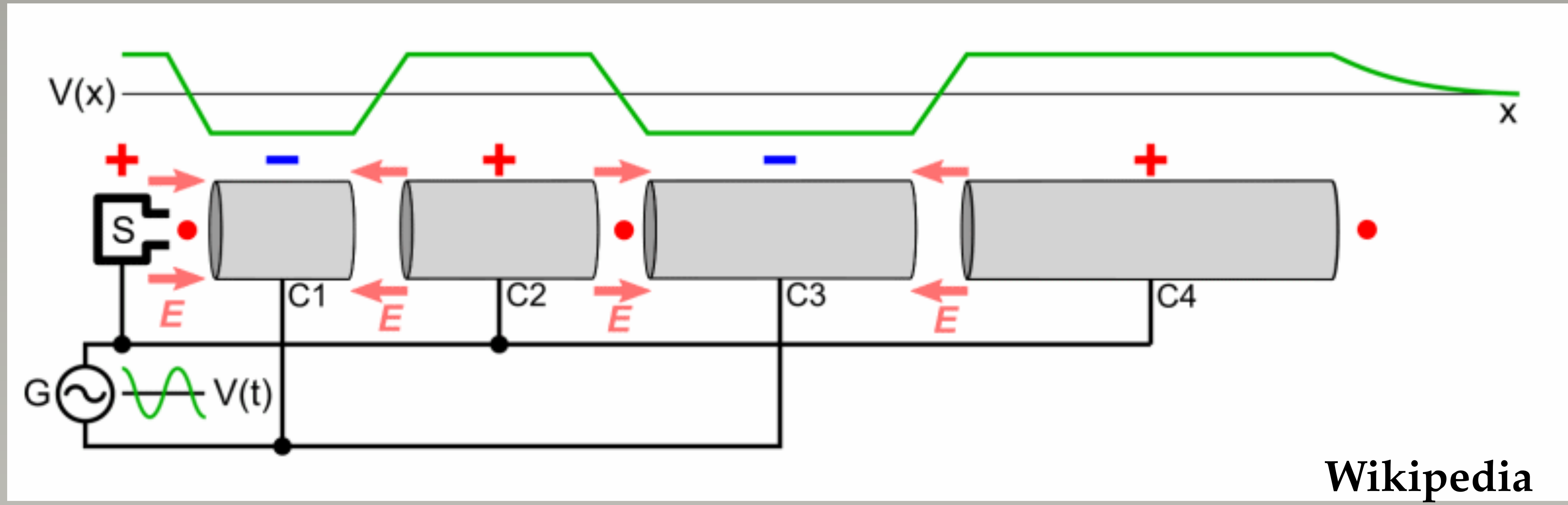

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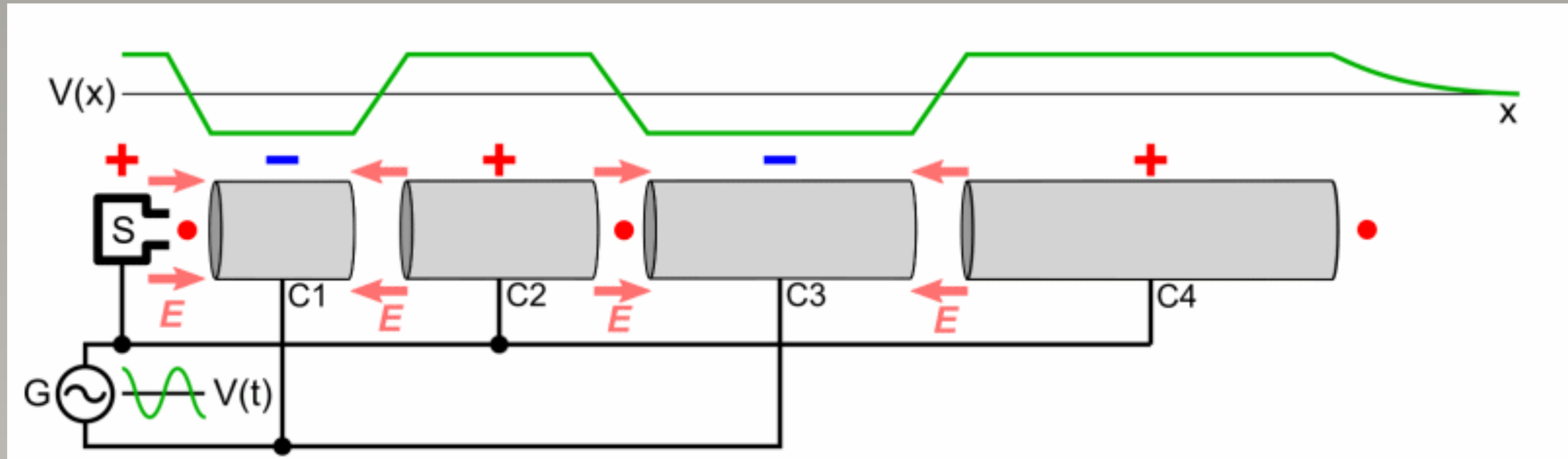

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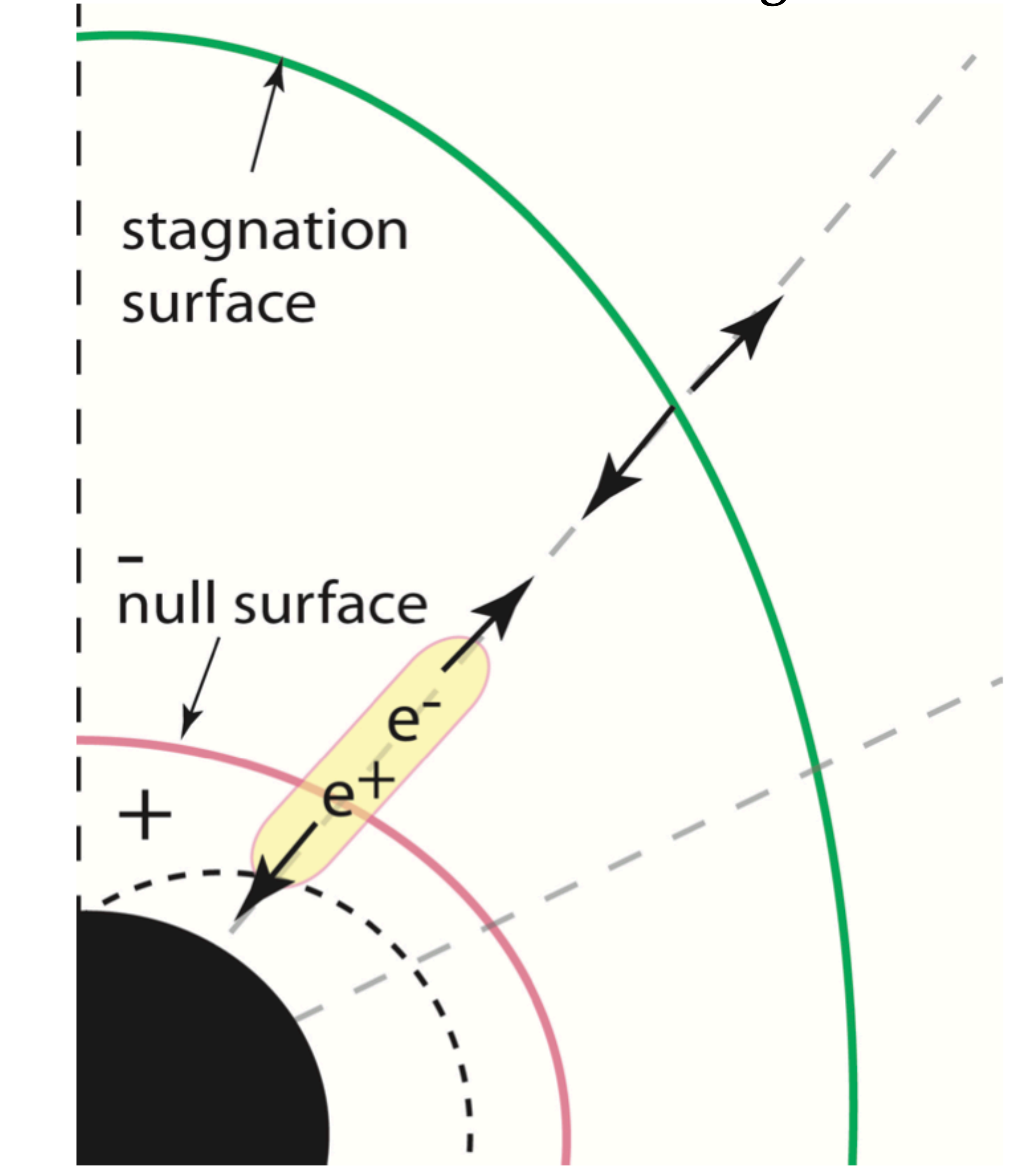
Gaps are also thought to form in the magnetospheres of compact objects.

Levinson & Segev 2018



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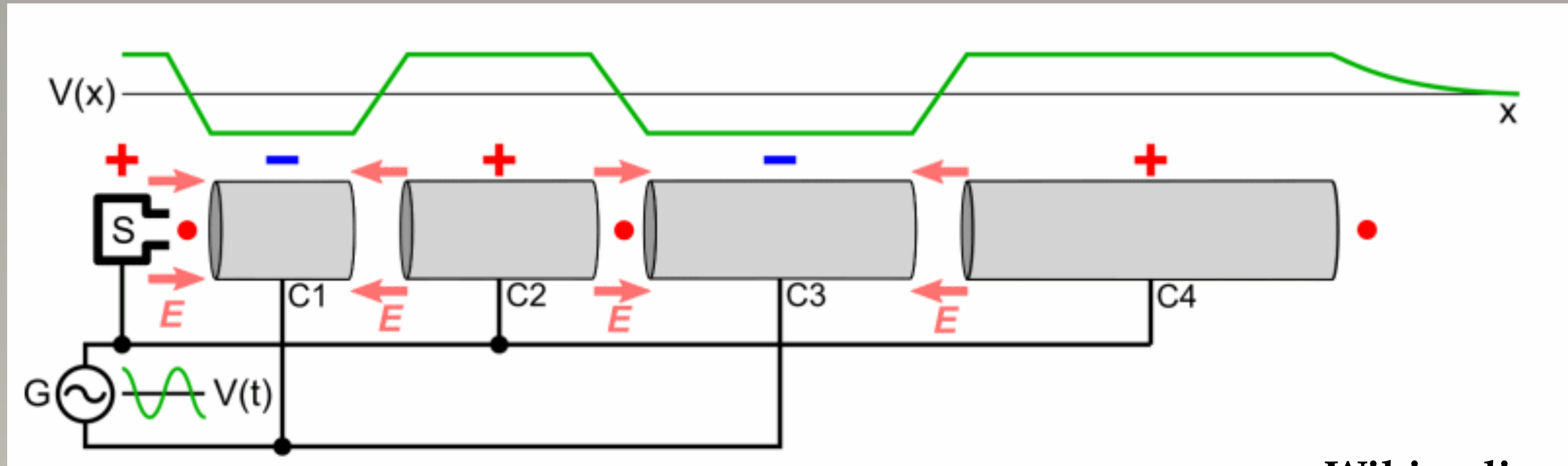

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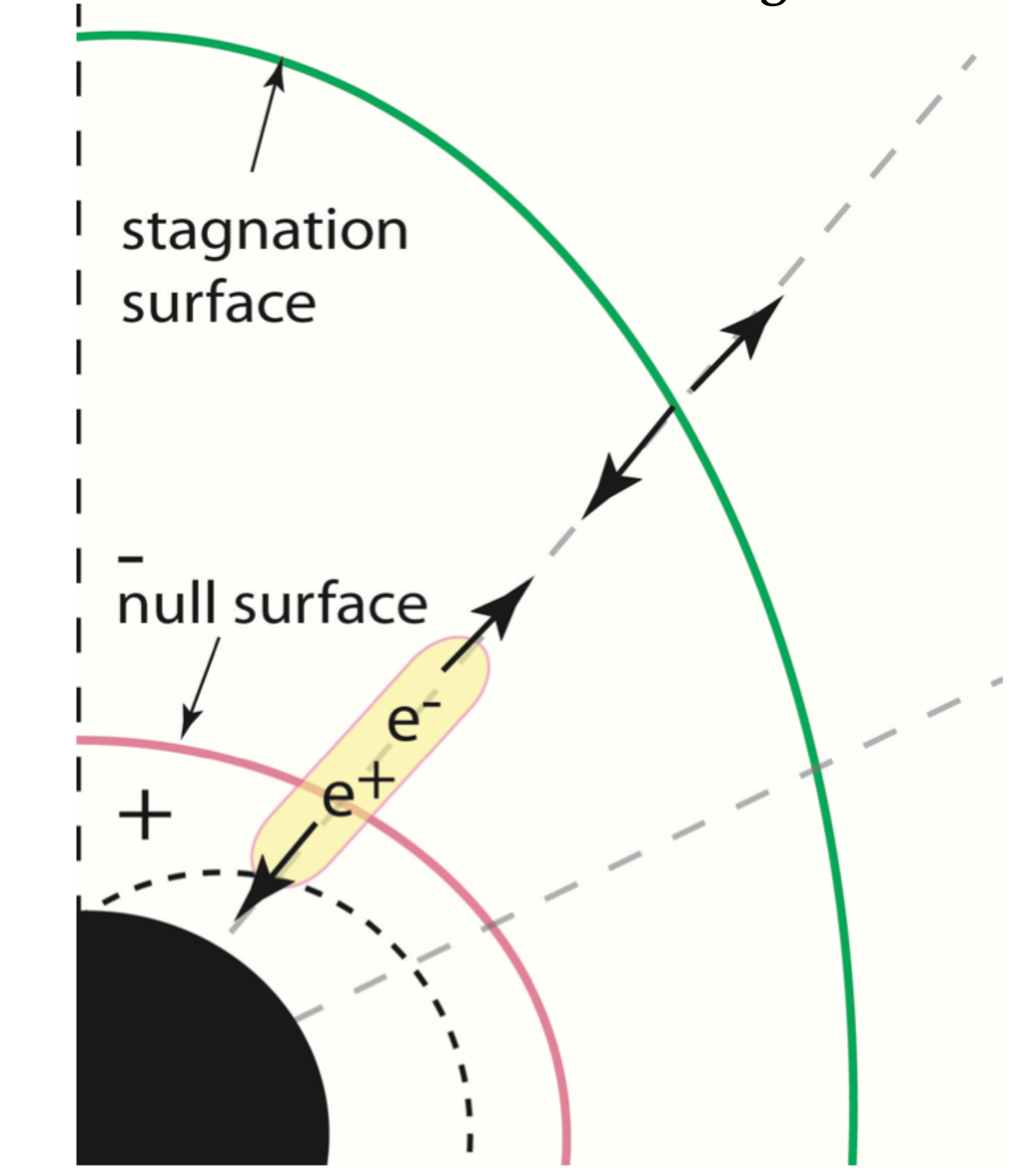

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Astrophysical sites with large scale regular E-fields:

- Gaps (pulsar and BH magnetospheres)
- Rotating fields of pulsars
- Oblique shocks
- Current sheets

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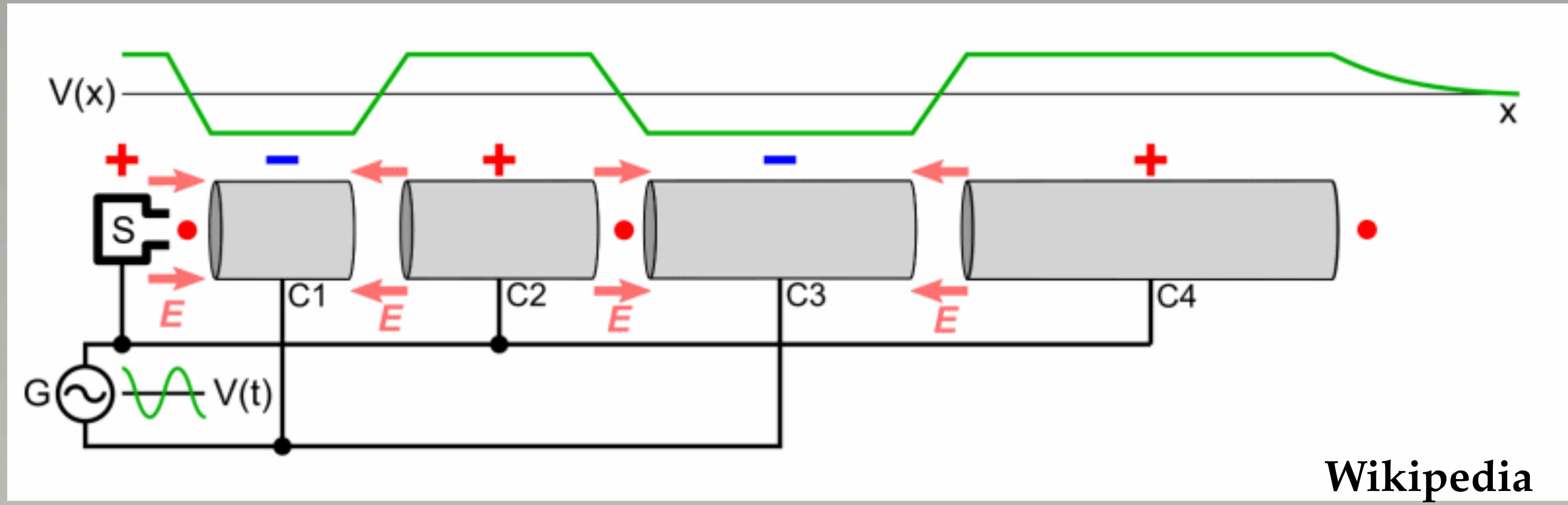

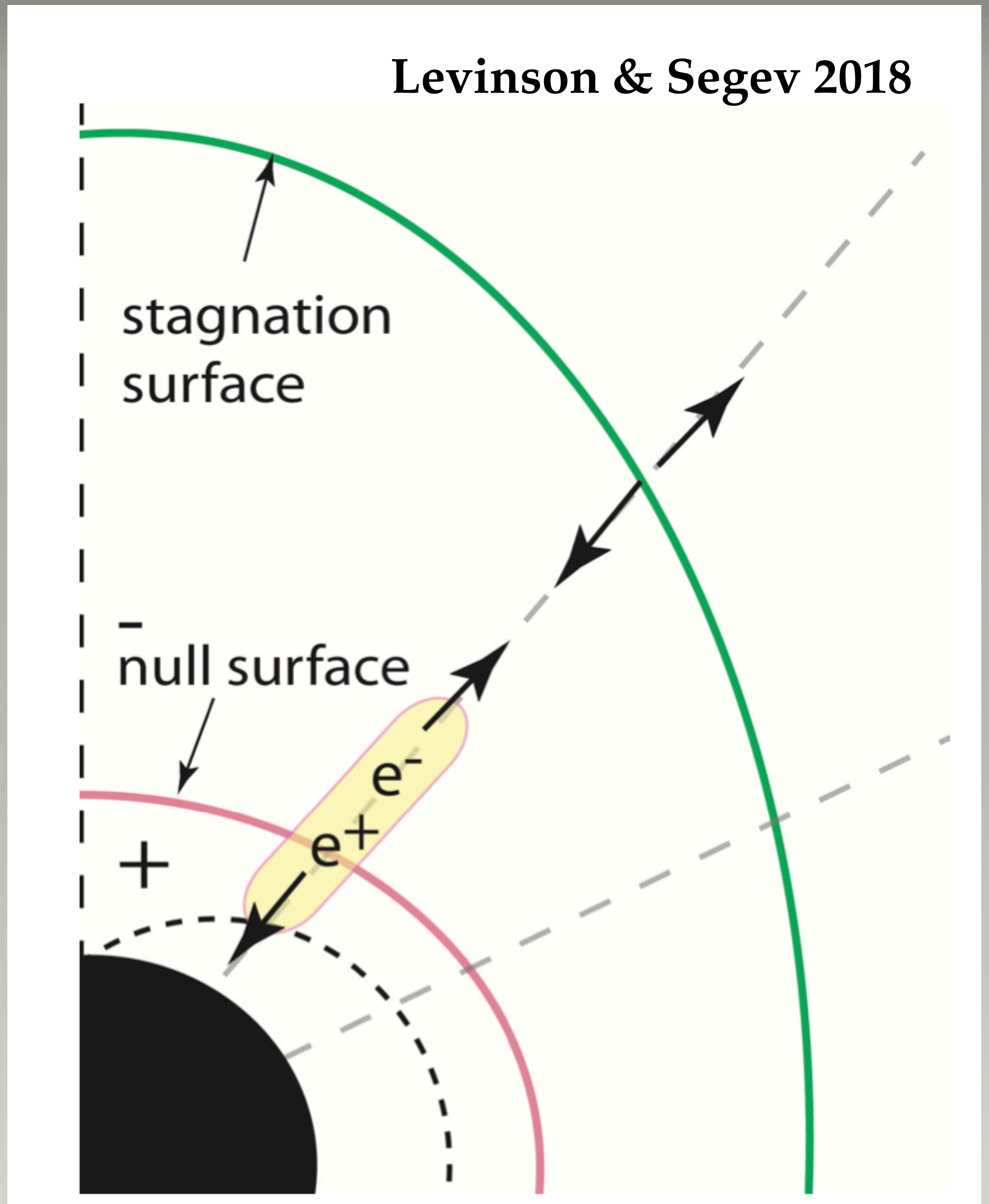

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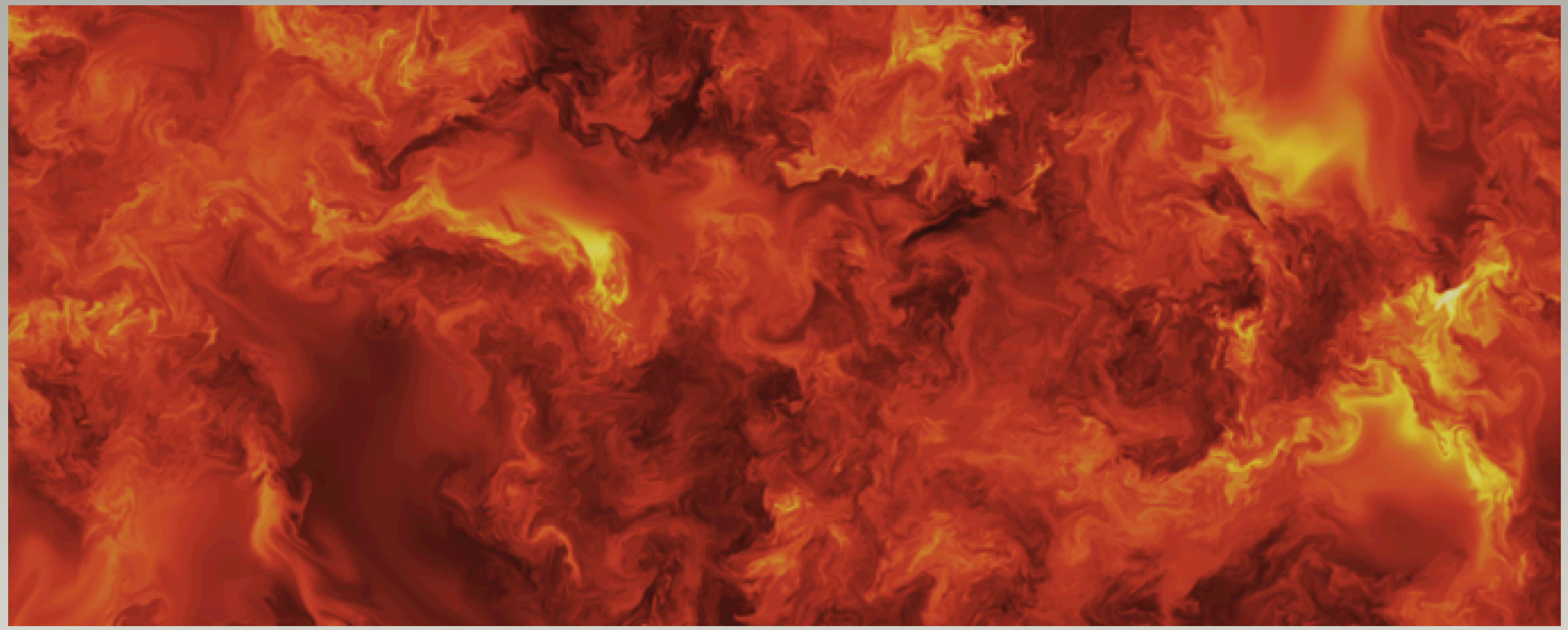
Regular E-fields accelerate all particles in the same way, either to the maximum system potential, or radiation reaction limit - How to produce power-laws?

Regular vs Stochastic Acceleration

In most systems, electric field is irregular.
For highly conducting plasmas (typical), a good approximation is: $\mathbf{E} = -\frac{1}{c}\mathbf{u} \times \mathbf{B}$



Credit: R. Meyrand

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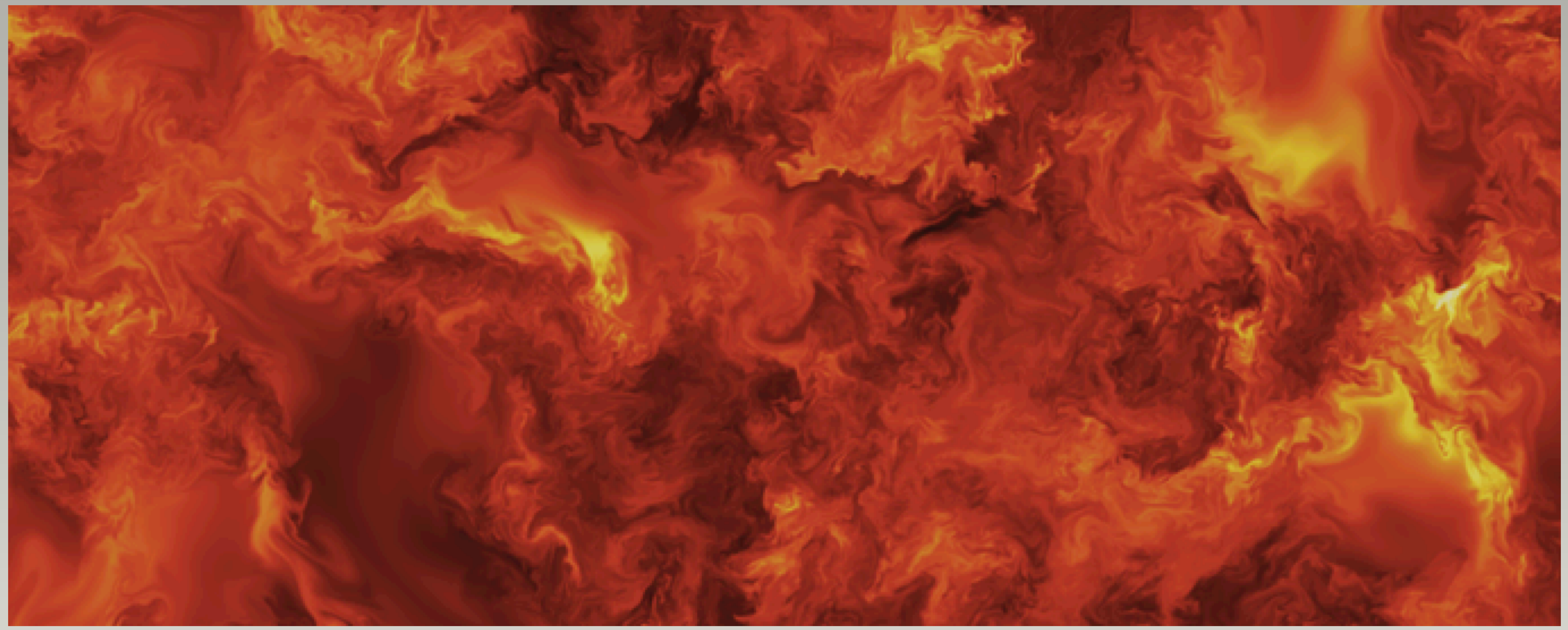

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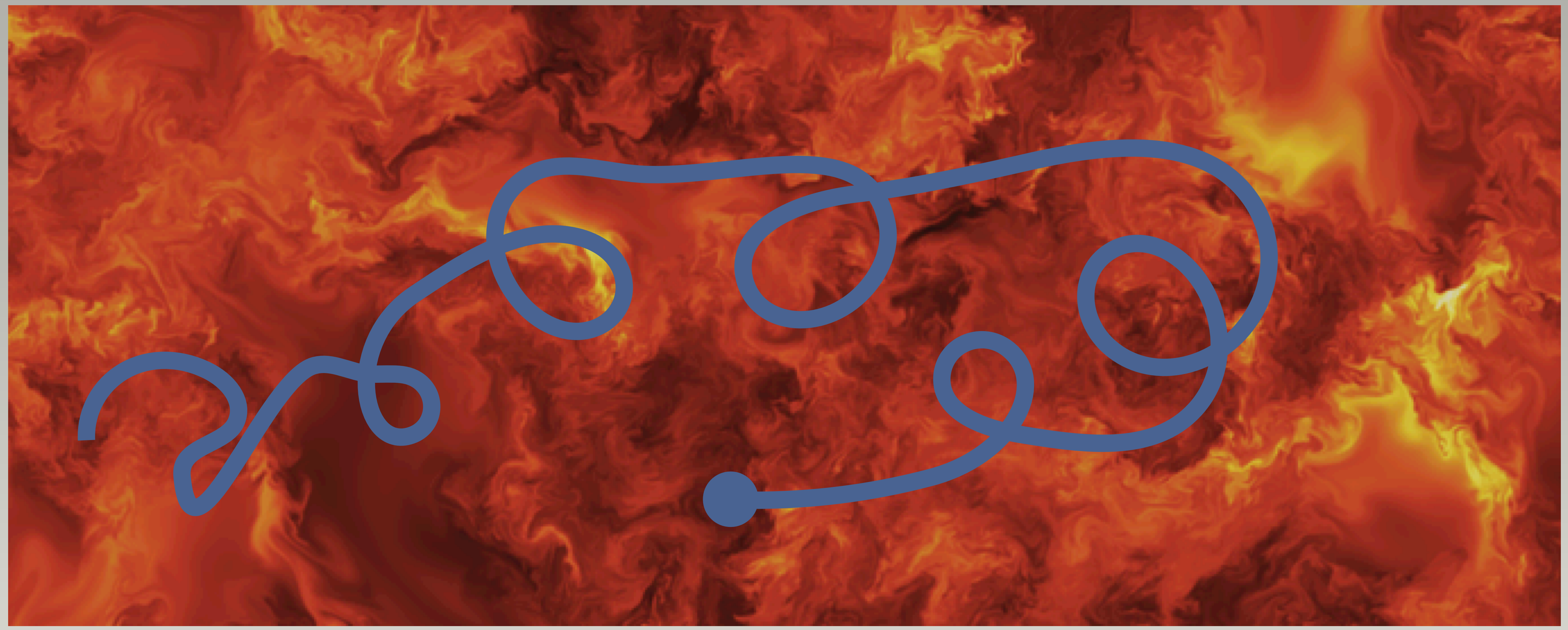
If particle can decouple from thermal gas, it makes a (not quite) random walk in momentum/energy space.



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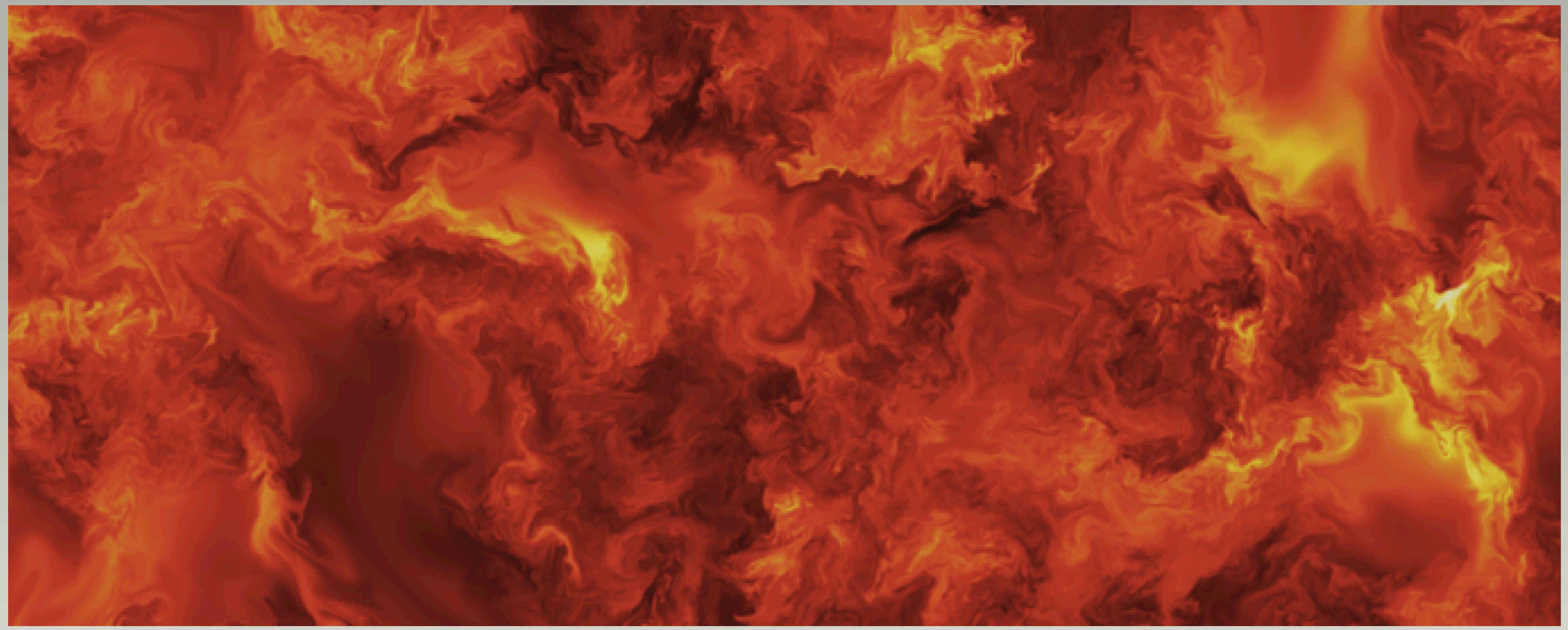
Consider scatterering centres of mass M, moving in random directions with speed U in some volume of radius R.

Test particles colliding with these clouds will try (but fail) to come into thermal equilibrium with $k_B T \approx MU^2$



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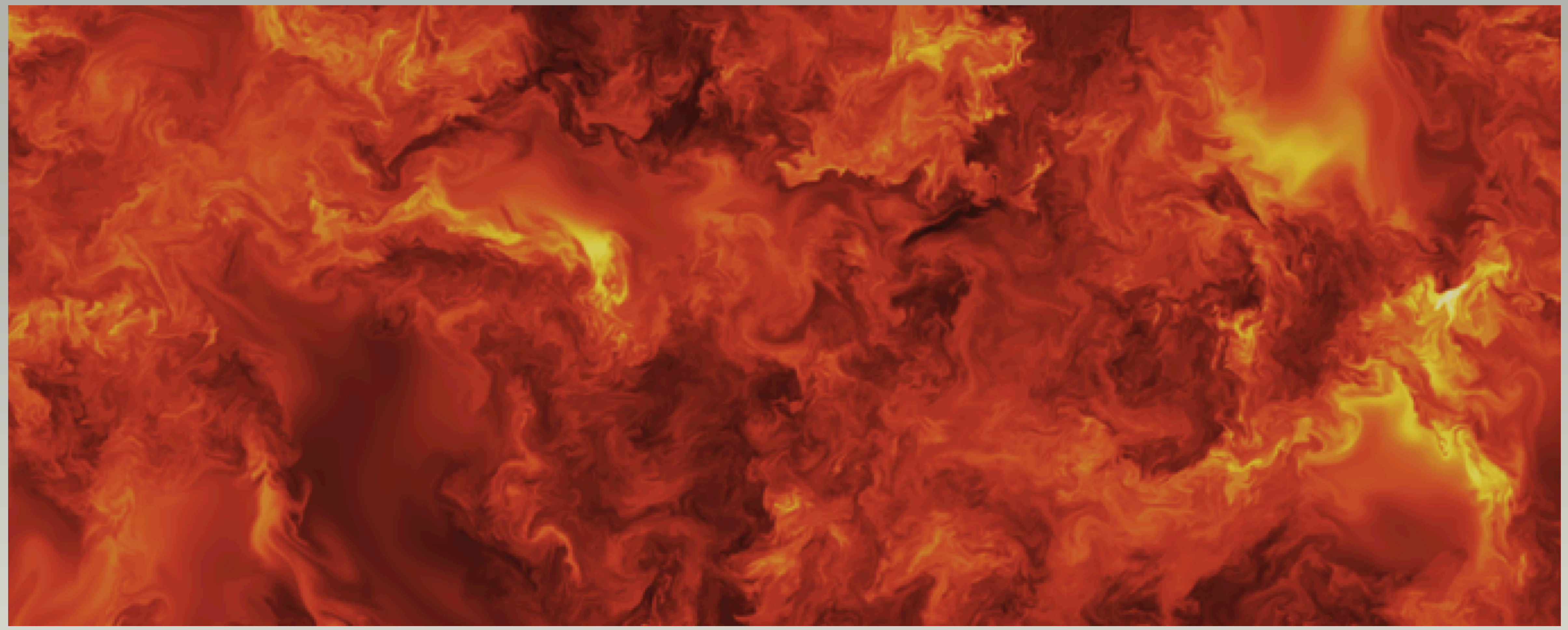

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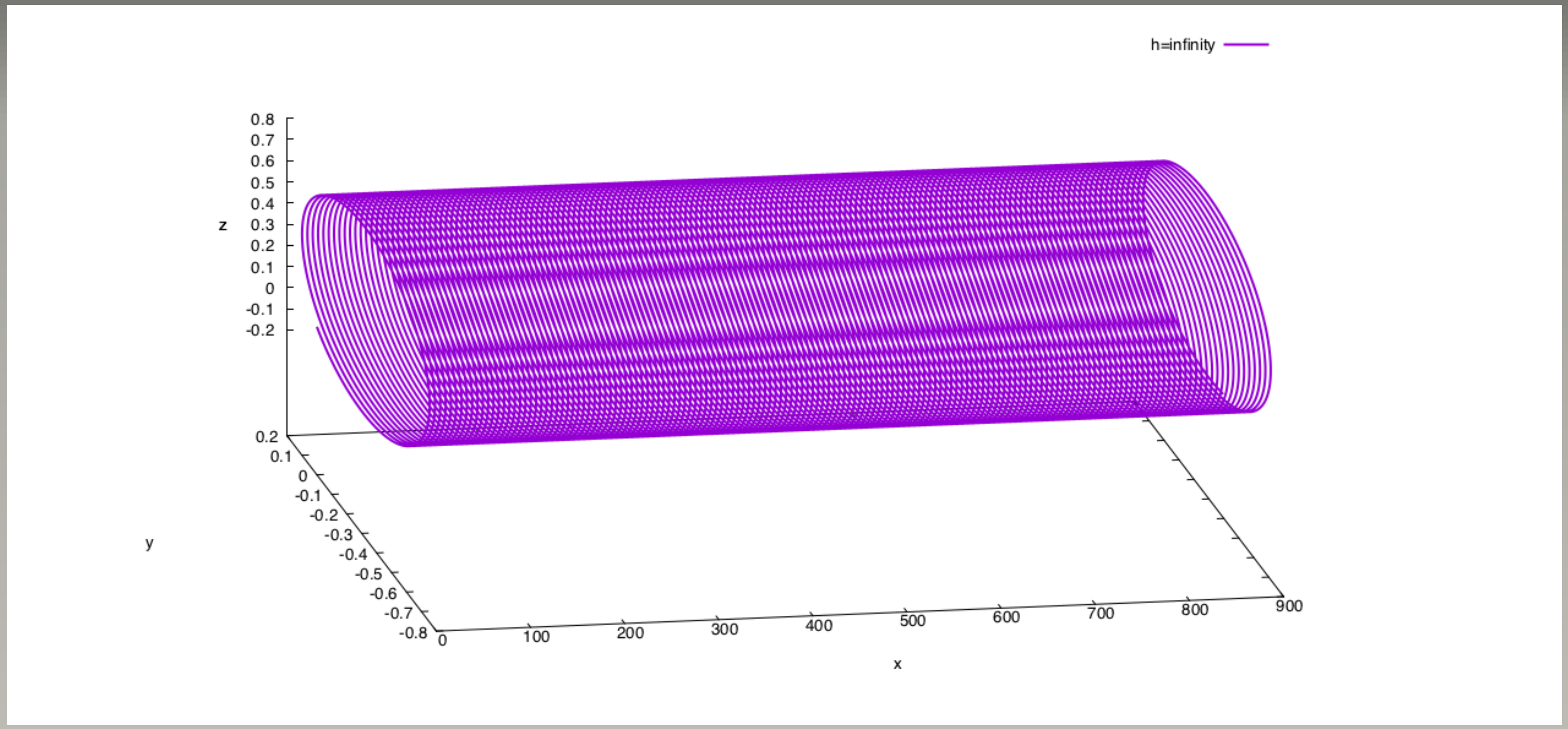
$$\varepsilon_{\max} = q \int \mathbf{E} \cdot d\mathbf{s} \approx q \frac{\bar{U}}{c} \bar{B} R = 10^{14} \frac{\bar{U}}{10\ \mathrm{km/s}} \frac{\bar{B}}{3\ \mu\mathrm{G}} \frac{R}{\mathrm{kpc}}\ \mathrm{eV}$$

Magnetised Transport

Astrophysical plasmas are typically <u>magnetised.</u>
This means particles undergoes "helical motion".

This requires

a) A large scale smooth magnetic field
b) Gyro-frequency >> Scattering rate*



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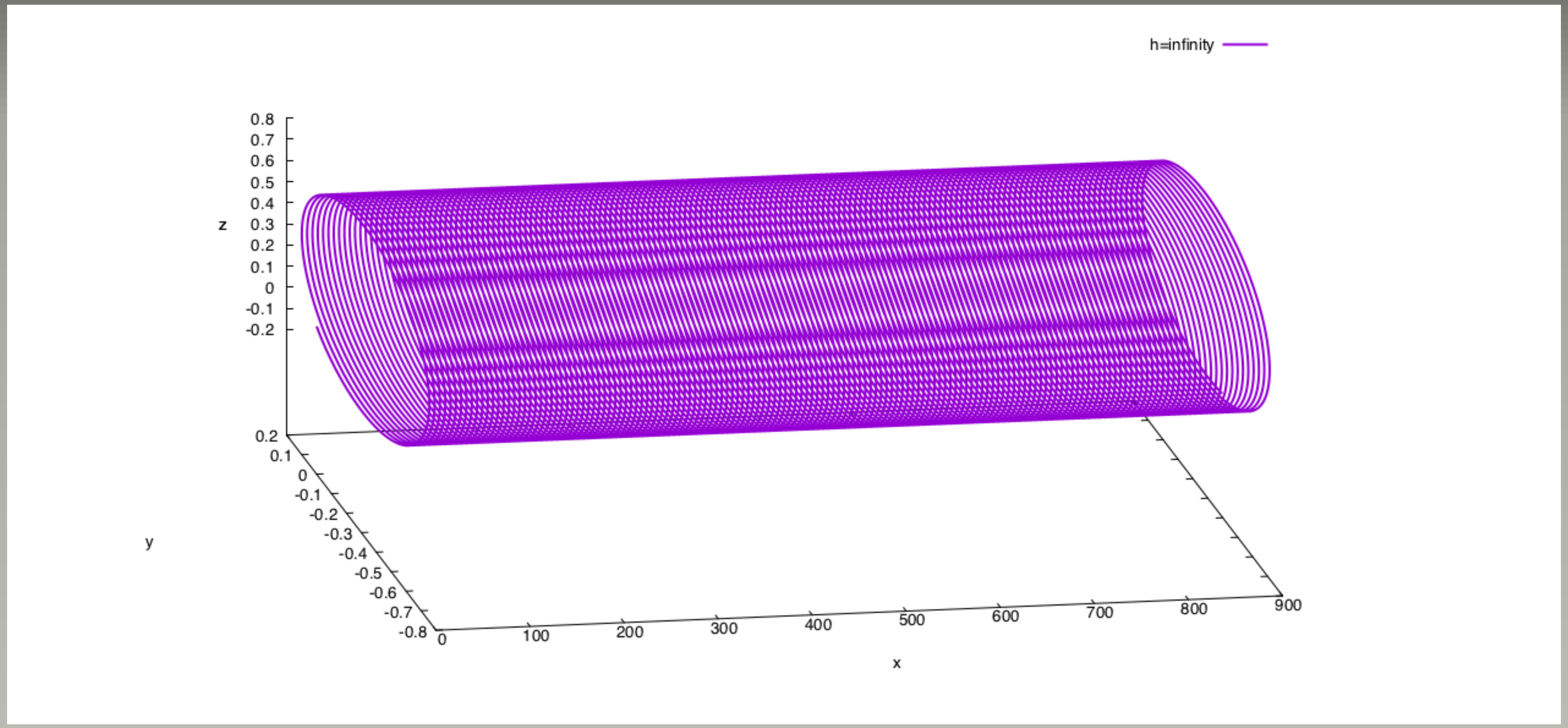

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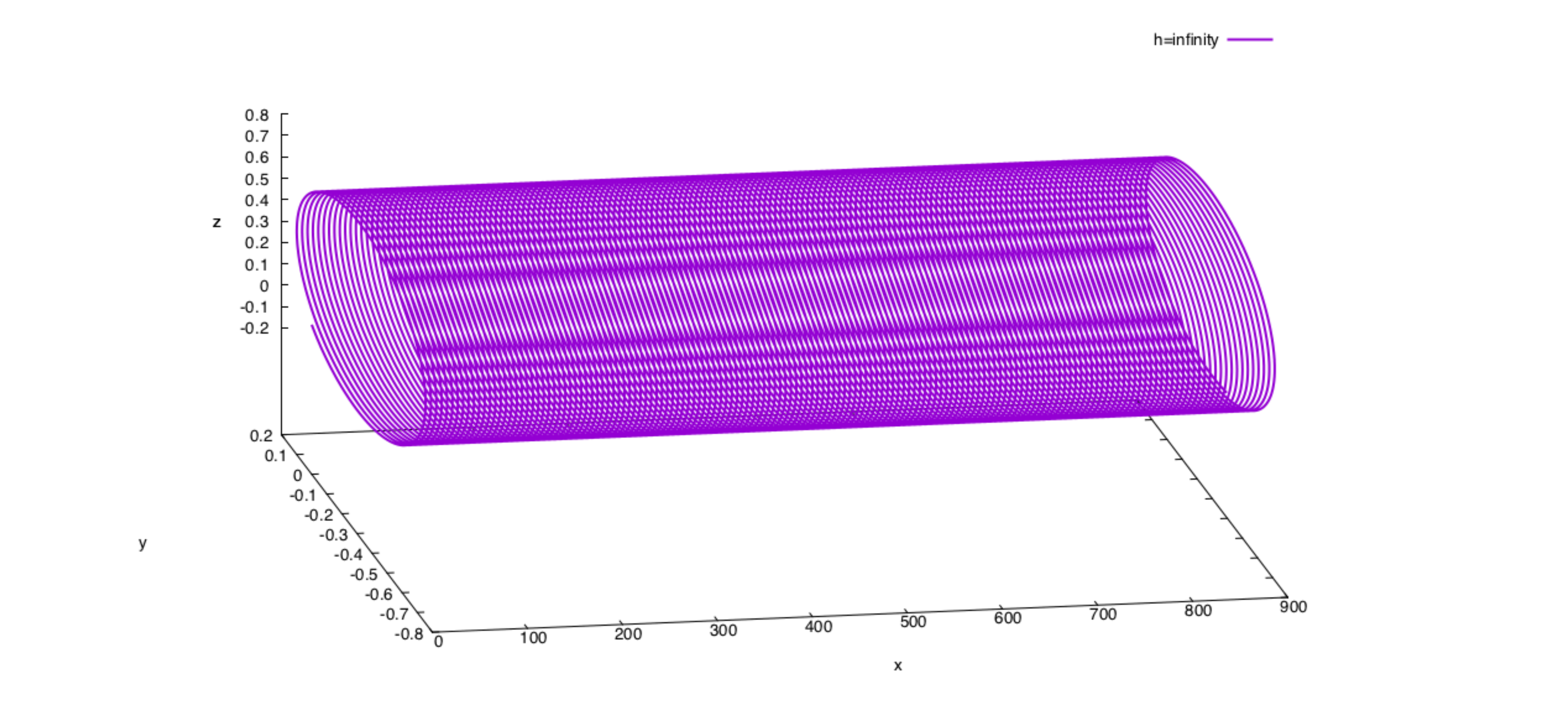

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Define, gyro-frequency and gyro radius

$$\omega_g \equiv \frac{qB}{\gamma mc} \quad \textbf{and} \quad r_g = \frac{v_\perp}{\omega_g}$$

and scattering frequency $\omega_g = h\nu_{sc}$

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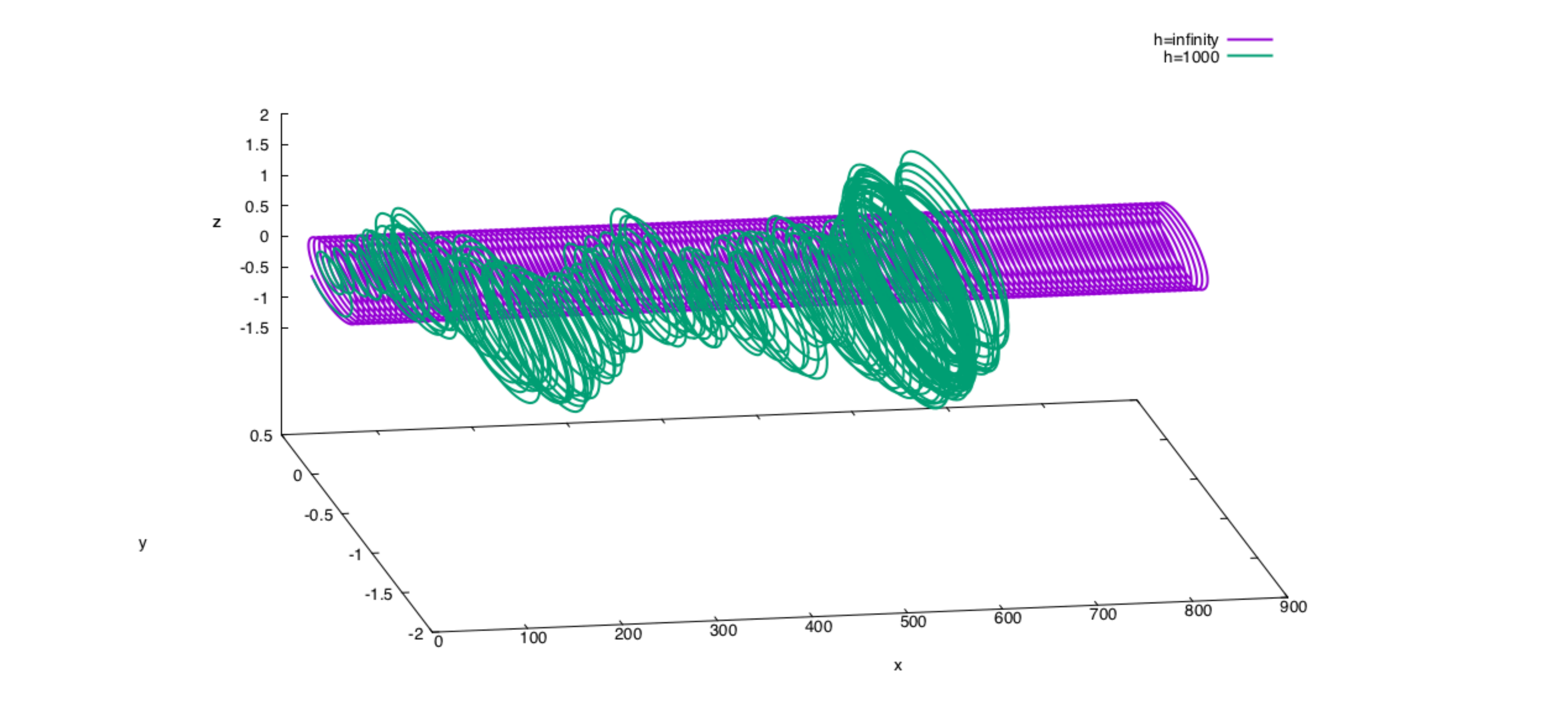

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Define, gyro-frequency and gyro radius

$$\omega_{\mathrm{g}} \equiv \frac{qB}{\gamma mc} \quad \textbf{and} \quad r_g = \frac{v_\perp}{\omega_g}$$

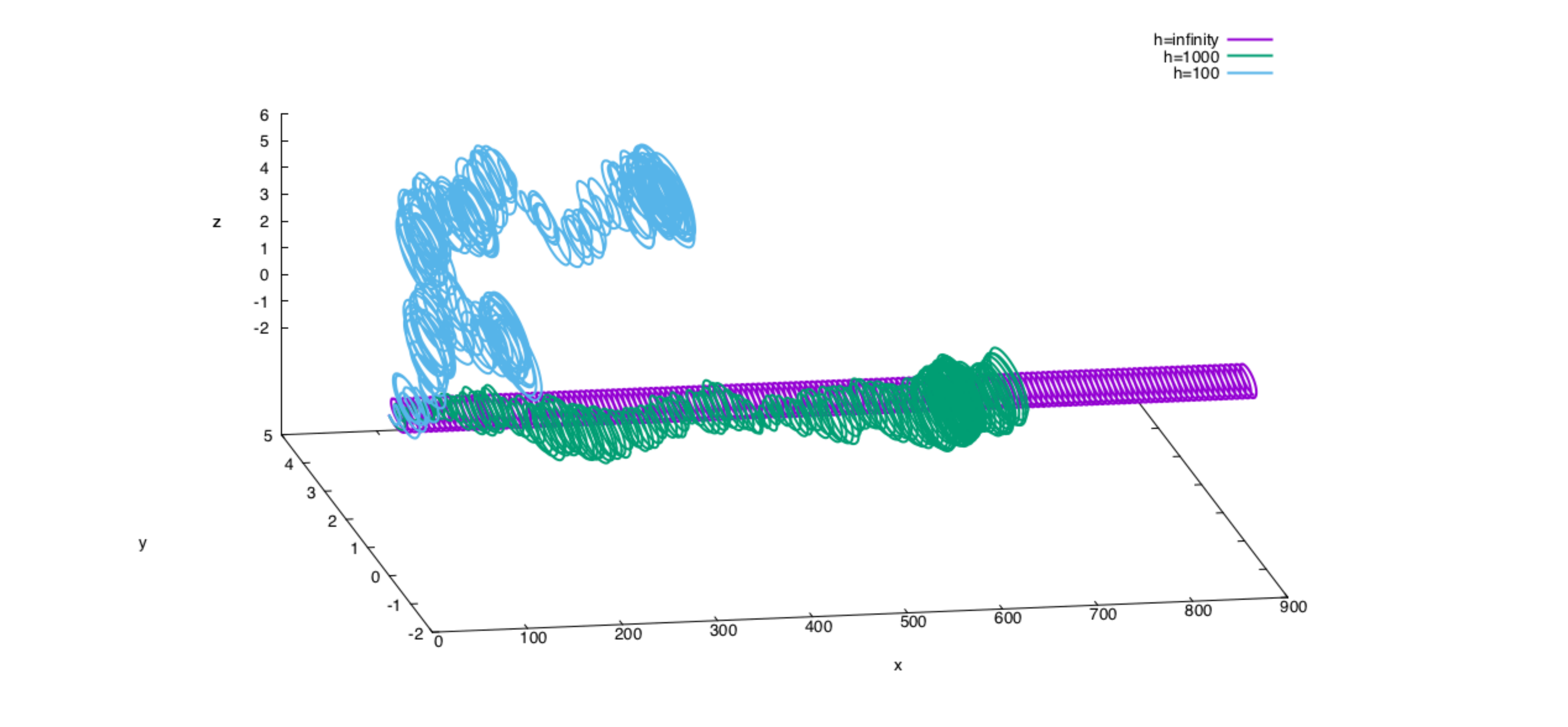
and scattering frequency $\omega_g = h\nu_{sc}$

*scattering is not by collisions with other particles but on quasi-stationary magnetic field fluctuations

Magnetised Transport

Astrophysical plasmas are typically <u>magnetised.</u>
This means particles undergoes "helical motion".

This requires
a) A large scale smooth magnetic field
b) Gyro-frequency >> Scattering rate*



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The particle distribution

It is typical when dealing with particles / physical kinetics to work with a distribution function:
A six dimensional quantity telling us how many particles are located between $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$, and between $\mathbf{p}$ and $\mathbf{p} + d\mathbf{p}$

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Mathematically we express this as $dN = f(\mathbf{p}, \mathbf{x}, t)\ d^3\mathrm{x} d^3\mathrm{p}$

In most (non-relativistic) sources, it is common to assume scattering maintains near isotropy, such that :

$f(\mathbf{p}, \mathbf{x}, t) = f_0(p, \mathbf{x}, t) + \delta f(\mathbf{p}, \mathbf{x}, t)$**, where** $\delta f \ll f_0$.

In such cases $dN \approx 4\pi p^2 f_0(\mathbf{p}, \mathbf{x}, t)\ d\mathrm{p}\ d^3\mathrm{x} = n(p, \mathbf{x}, t)\ d\mathrm{p}\ d^3\mathrm{x}$

Key Points

- **Only an electric field can do work on a particle**
- **Electric field vanishes in local fluid frame:** $\mathbf{E} = -\frac{1}{c}\mathbf{u} \times \mathbf{B}$
- **Theoretical energy limit given by potential across system**

$$\varepsilon_{\max} = q\int \mathbf{E}\cdot d\mathbf{s} \approx q\frac{\bar{U}}{c}\bar{B}R = 10^{14}\frac{\bar{U}}{10\ \mathrm{km/s}}\frac{\bar{B}}{3\ \mu\mathrm{G}}\frac{R}{\mathrm{kpc}}\ \mathrm{eV}$$

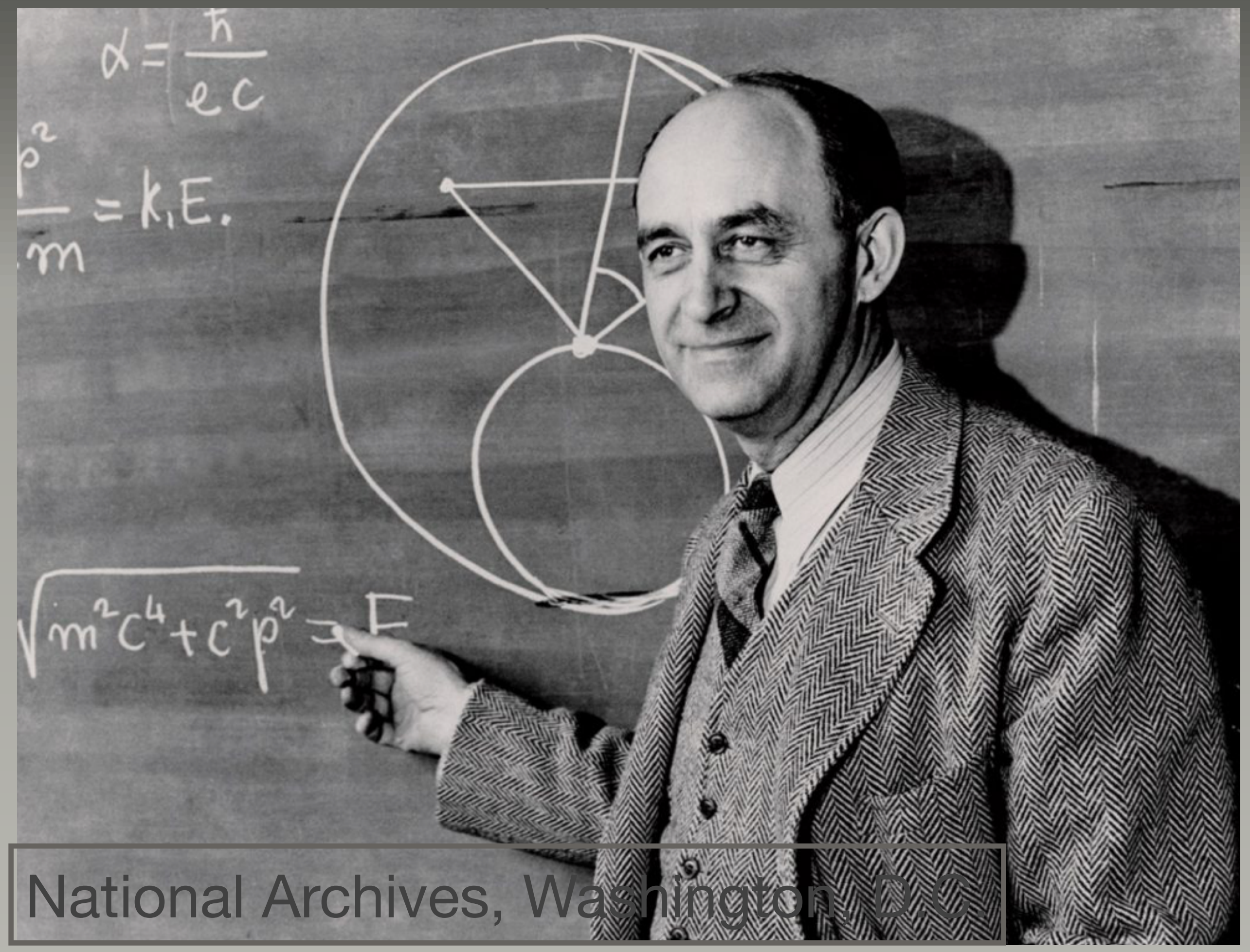
This is the Hillas limit. It can be conveniently expressed as $r_g = (u/c)R$

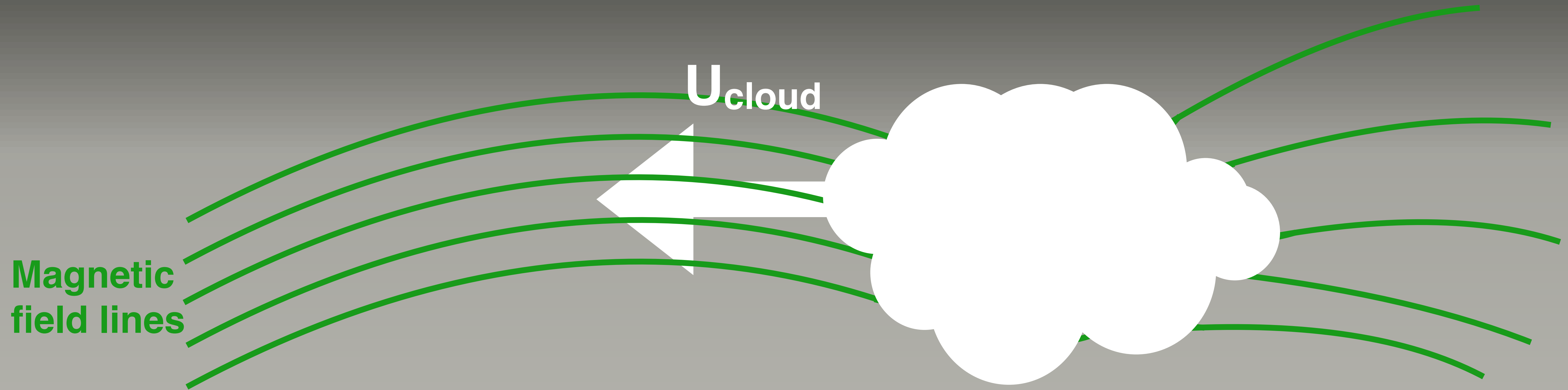
- **Particles are typically magnetised, and to lowest order follow magnetic field lines on helical trajectories**
- **Particles undergo regular scattering on magnetic field fluctuations**
- **On scales >> mean free path** $\lambda \equiv \mathrm{v} / \nu_{\mathrm{sc}}$ **particles are approximately isotropic**

Lecture Overview

- Non-thermal emission from astrophysical systems
- Particle acceleration essentials
- **Enrico Fermi's great insight**
- Diffusive Shock Acceleration
- A quick digression into plasma physics

Fermi Acceleration in a Nutshell





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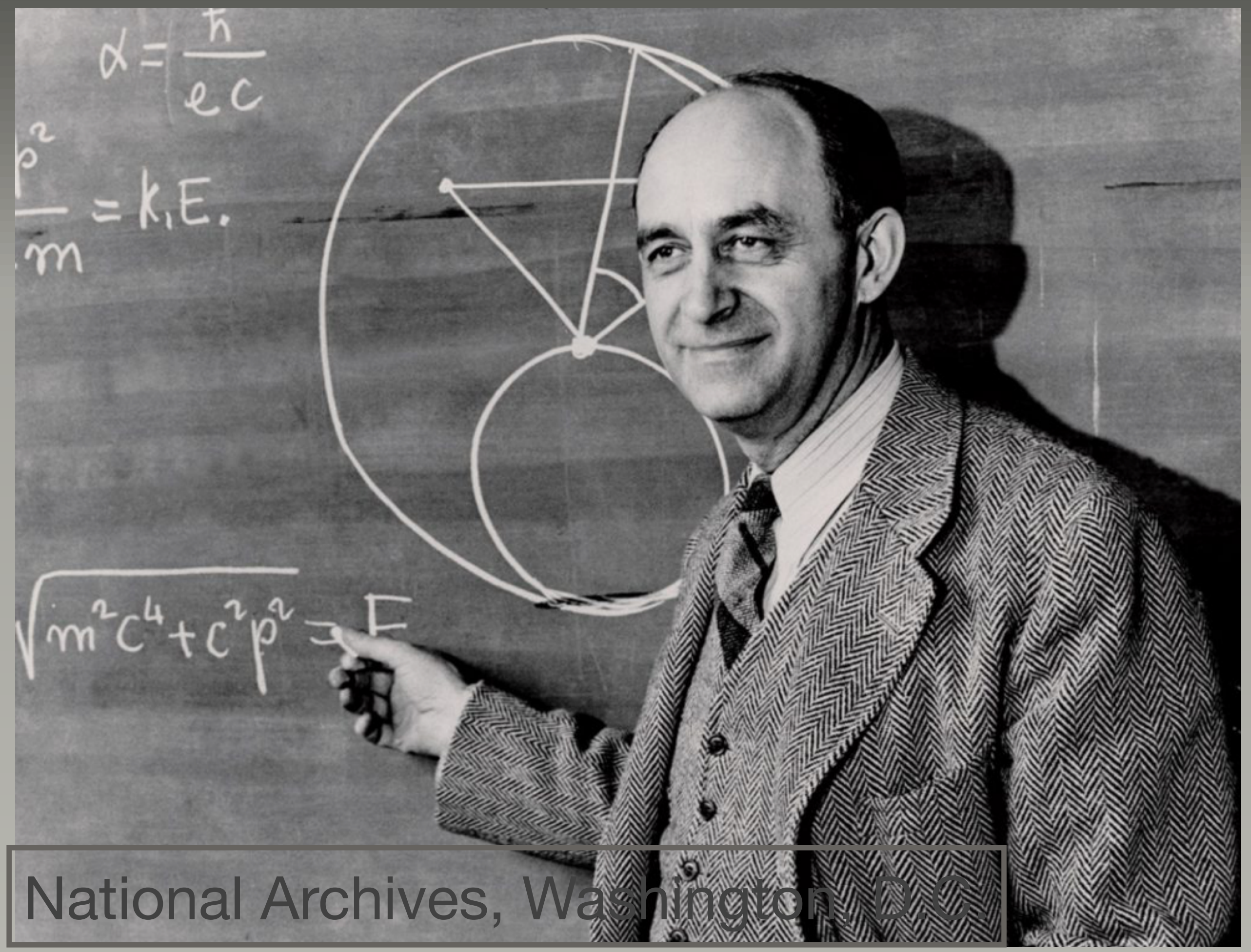

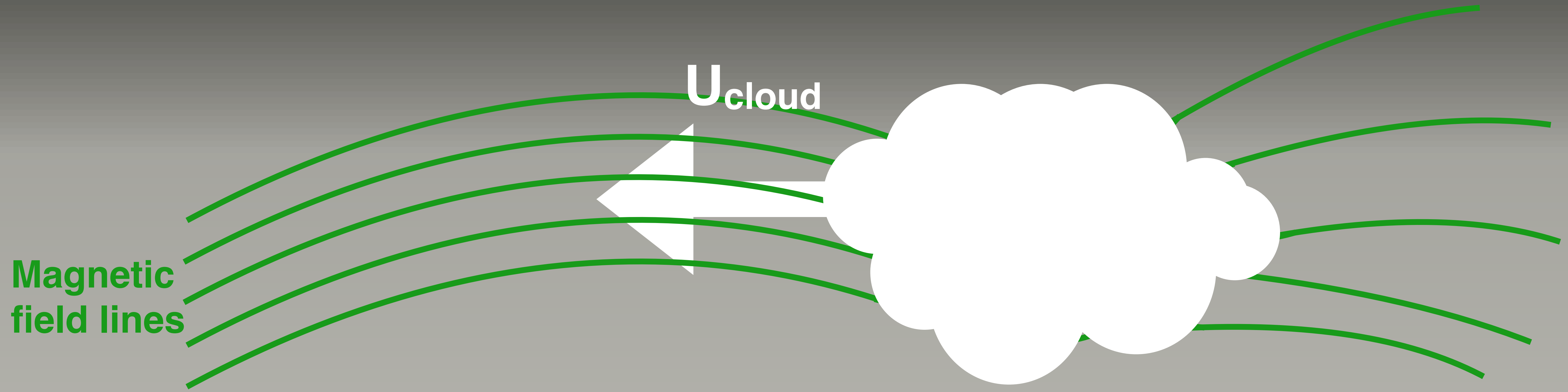

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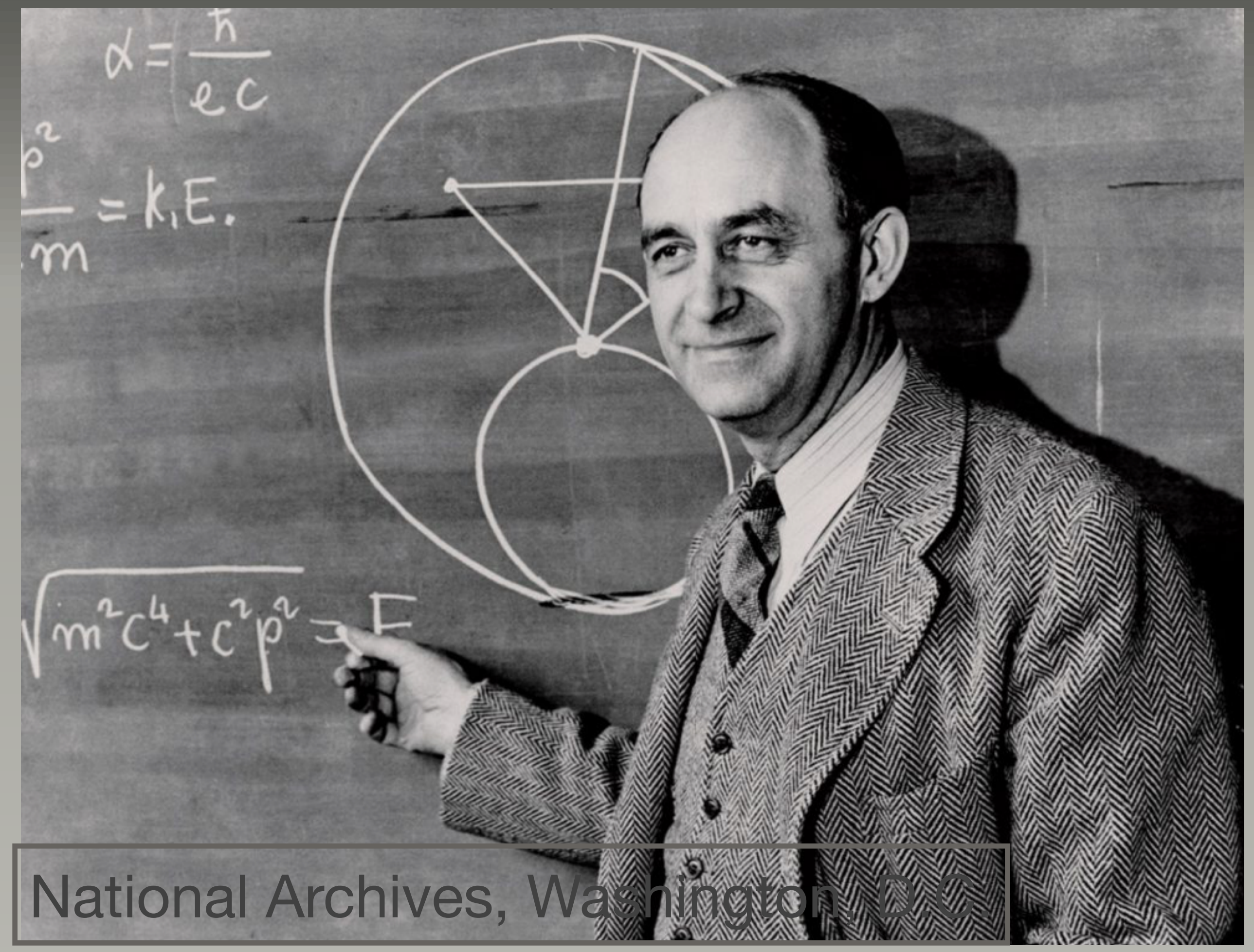
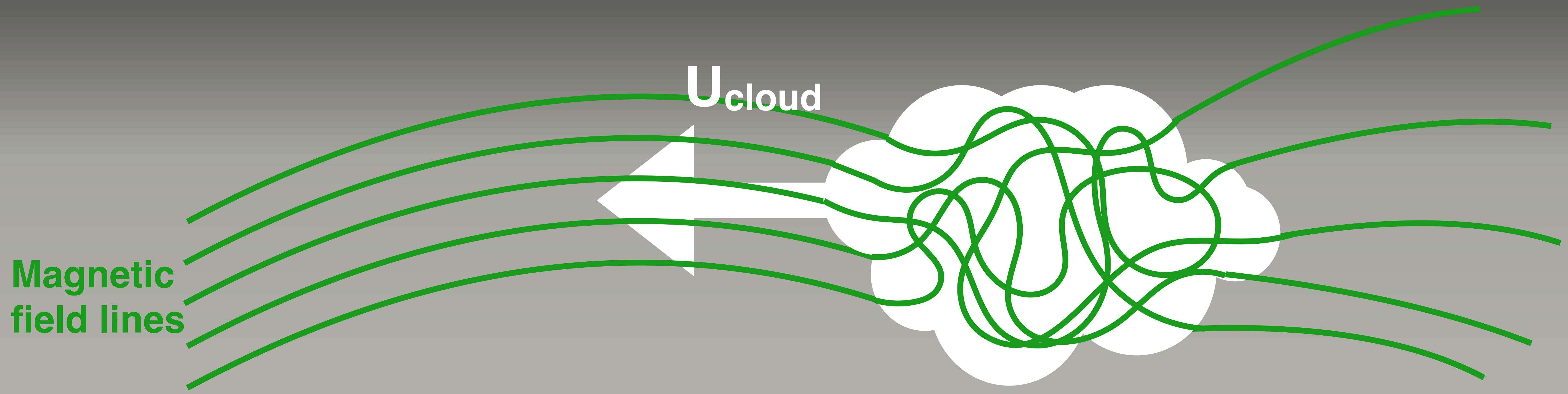
Fermi considered particles with $\mathrm{v} \gg \langle u \rangle$ which can sample different fluid velocities.

In Fermi's original picture he exploited the fact that the ISM was magnetised, and filled with High-Velocity Clouds moving at ~10 km/s.

As we saw previously, particles "try" to come into thermal equilibrium with such clouds. <u>But how?</u>

Fermi Acceleration in a Nutshell





Consider particle with initial velocity v (>> U_{cloud}), and momentum p.

We perform a Galilean transformation to the frame of the cloud: $\mathbf{p}'_{\mathrm{init}} = \mathbf{p}_{\mathrm{init}} - m\mathbf{U}_{\mathrm{cloud}}$

Let's restrict ourselves to 1 dimension, such that $|\mathbf{p}'_{\mathrm{init}}| = p_{\mathrm{init}} - mU_{\mathrm{cloud}}$

Energy approximately conserved in frame of cloud, but can be scattered/mirrored.

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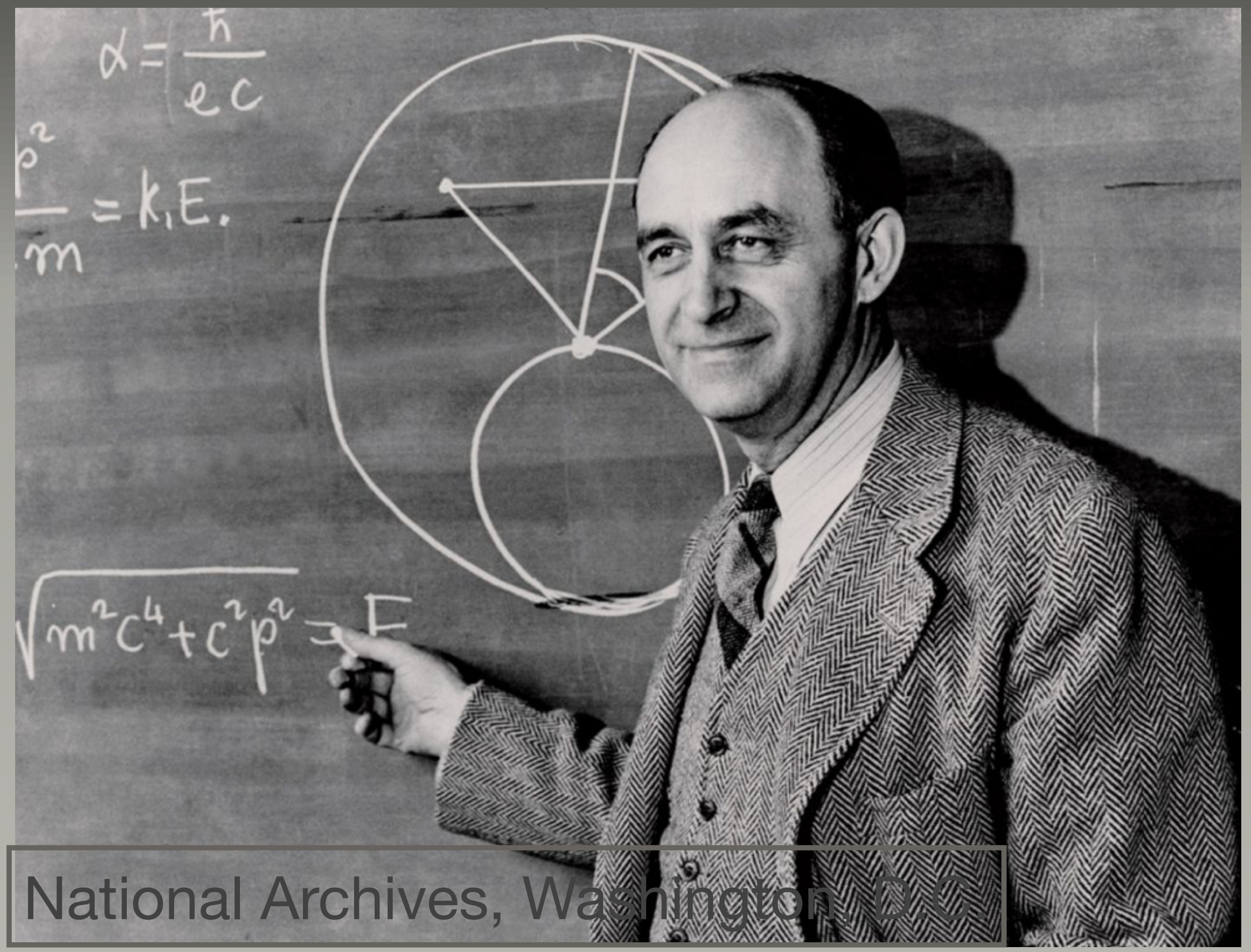

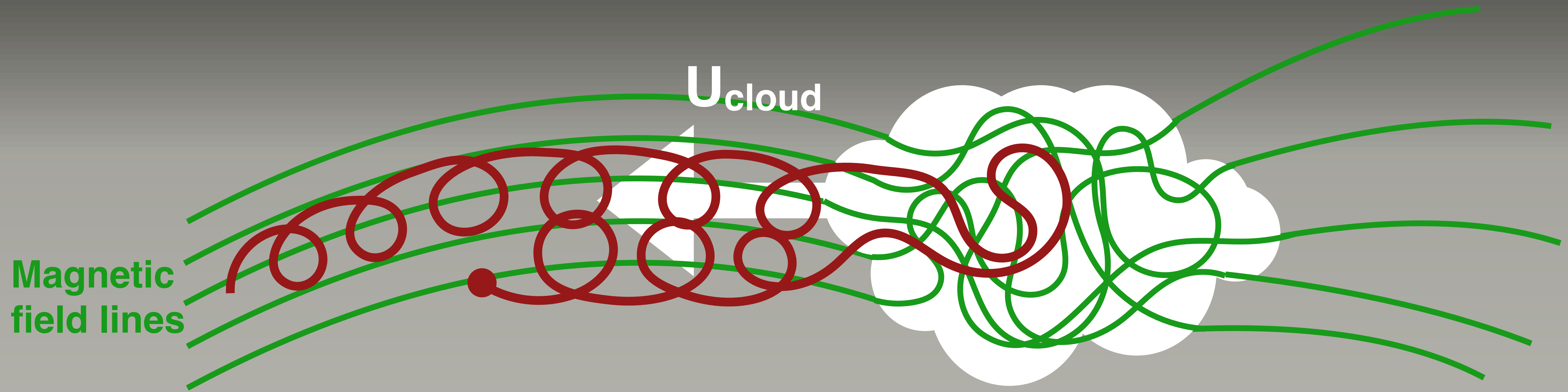

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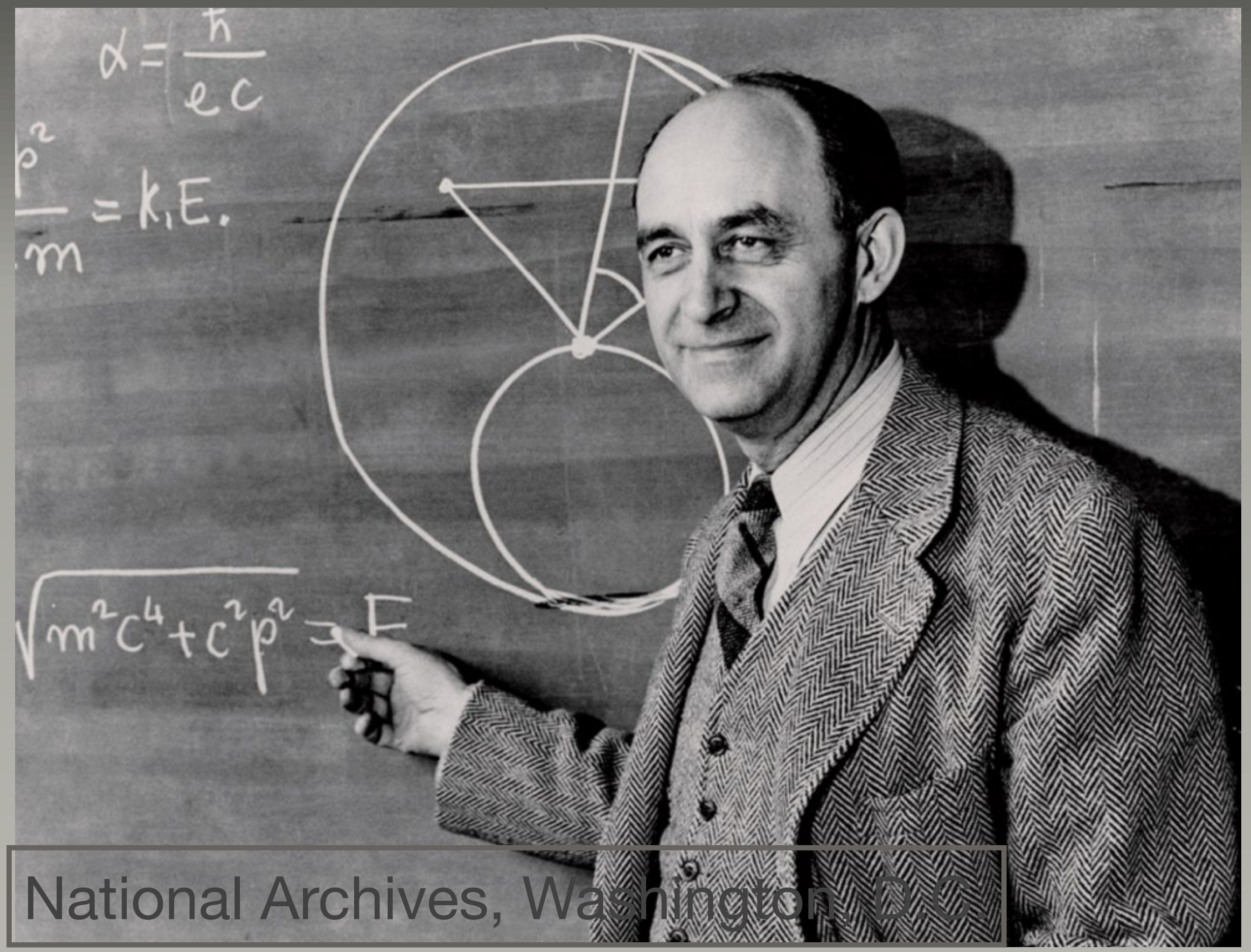

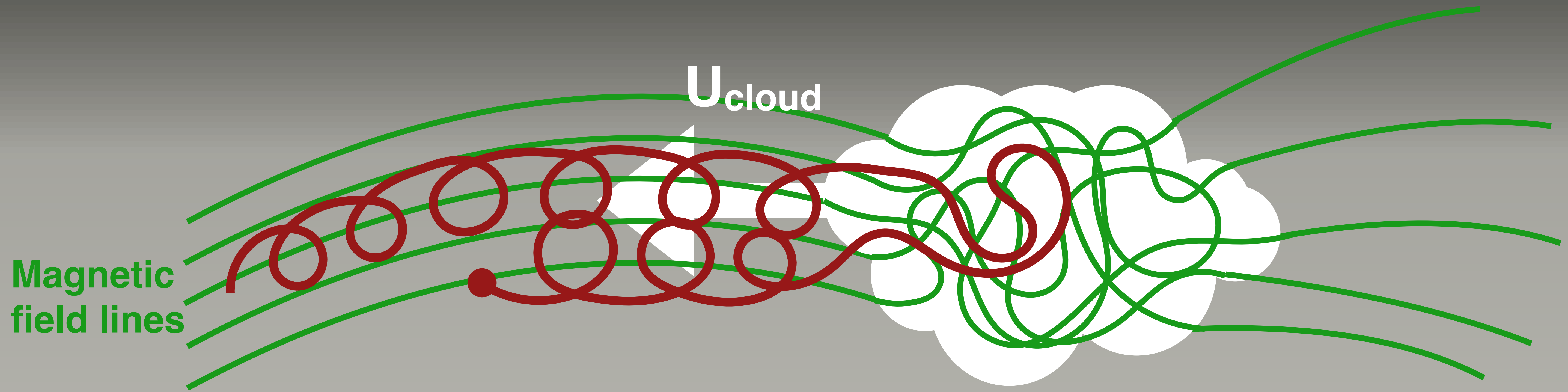

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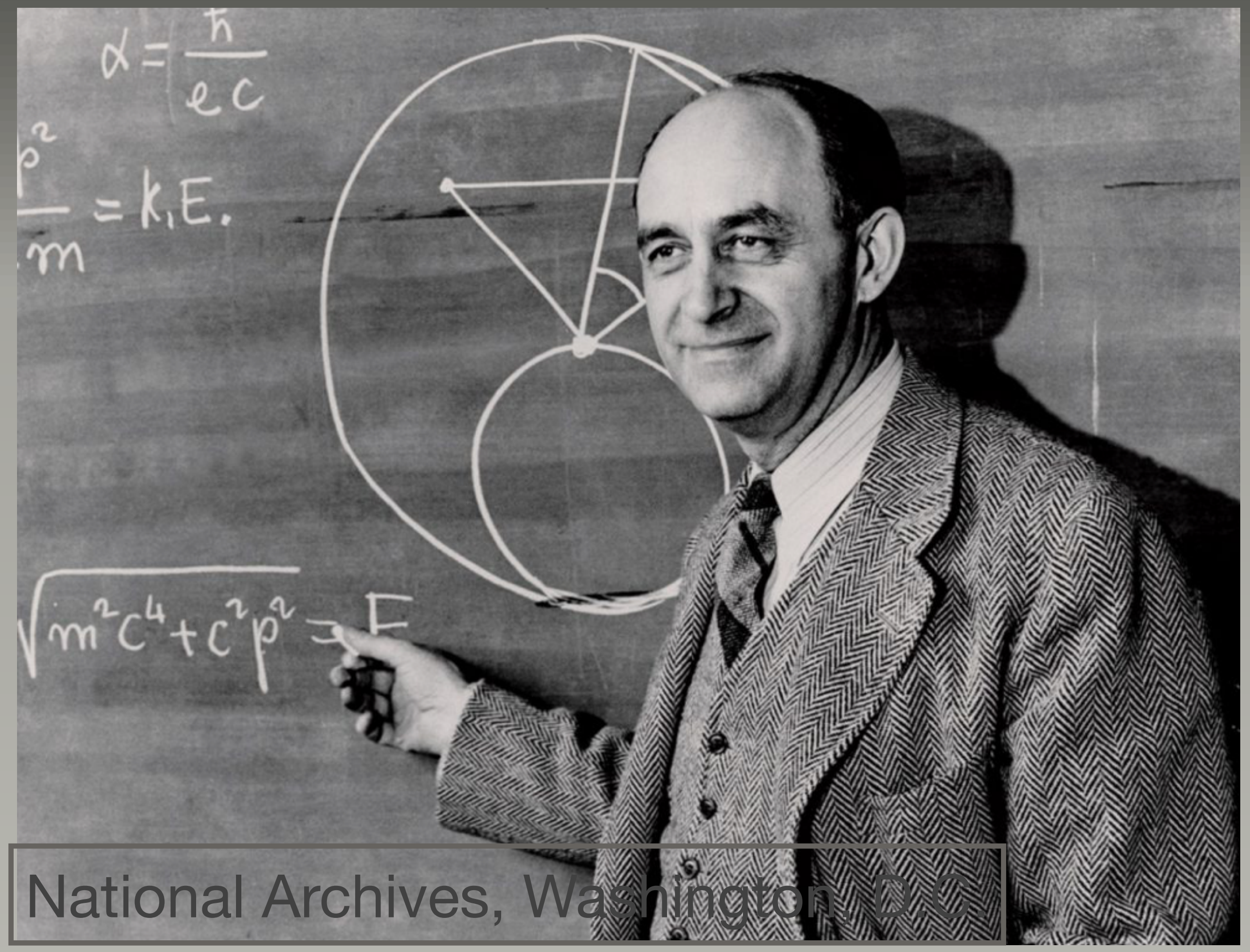

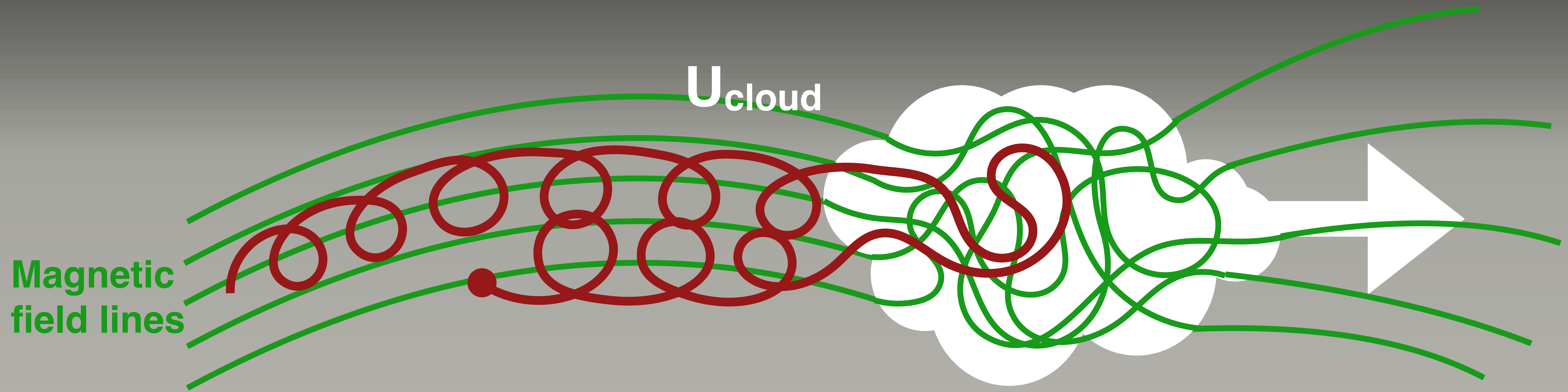

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-> for relativistic particles $\left\langle \frac{dp}{dt} \right\rangle = \alpha p$

Power-laws?

Particles gain momentum at rate $\dfrac{\langle \Delta p \rangle}{\Delta t} = \alpha p$ **where** $\alpha \propto (u/v)^2$

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Solving: $\dfrac{\partial N_>(p)}{\partial p} \Delta p = -\dfrac{\Delta t}{t_{\rm esc}} N_>(p)$ **or** $\dfrac{\partial \ln N_>(p)}{\partial \ln p} = -\dfrac{1}{\alpha t_{\rm esc}}$

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Power laws "possible" if $t_{\rm esc}$ **is energy independent (it usually isn't!) or is incredibly large (no escape,** $n(p) \propto p^{-1}$**)**

Key Points

- Fermi established a mechanism to “energise” particles.
  (although they need to energetic already to participate in first place!)

- We refer to any mechanism where by particles are accelerated by sampling relative motion of fluids as <u>Fermi acceleration</u>

- The average energy change scales as the square of the velocity, hence we call it second order Fermi acceleration

- Power-laws possible, but requires some fine tuning.

- Modern approach uses scattering fluctuations in plasma instead of clouds. Conclusions are very similar

Lecture Overview

- Non-thermal emission from astrophysical systems
- Particle acceleration essentials
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- **Diffusive Shock Acceleration**
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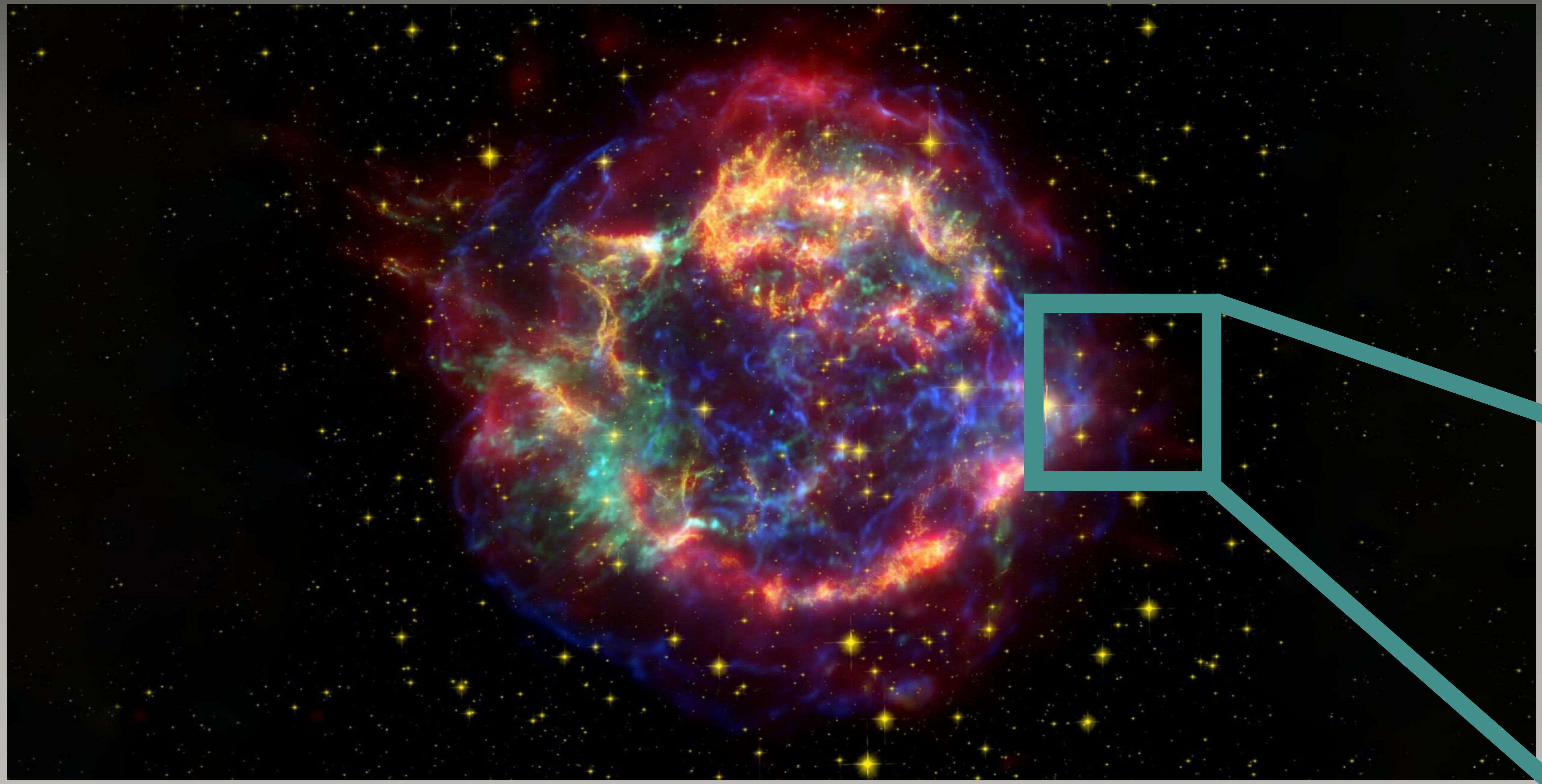
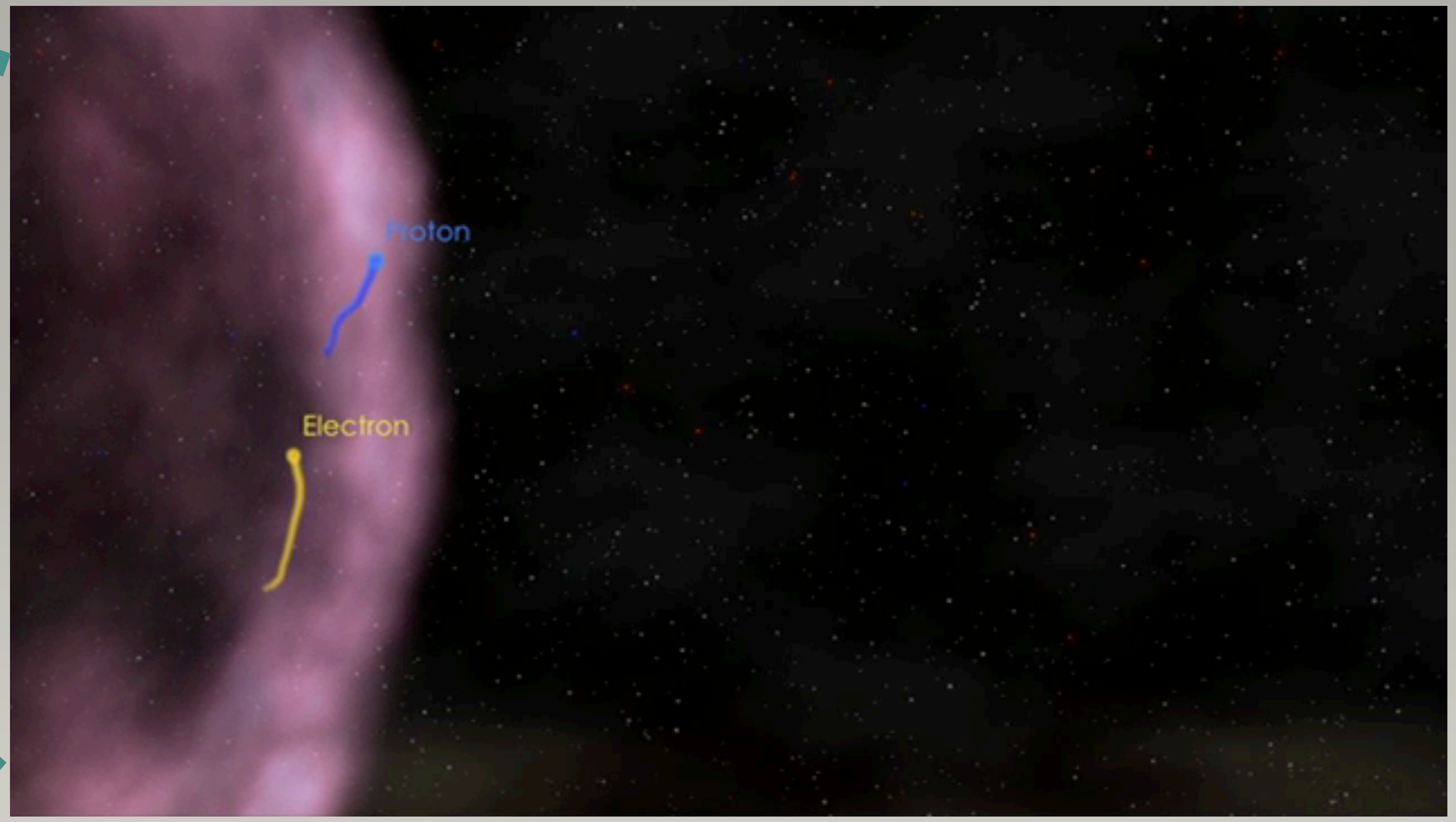
Particle acceleration at shocks



Focus on shocks with velocity << c.

Shocks produce a jump in the flow velocity across a narrow layer



Theory developed in late 70s independently by 4 different groups, Krymskii 77, Blandford & Ostriker 78, Axford, Leer & Skadron 77, Bell 78

Image and movie credit : NASA

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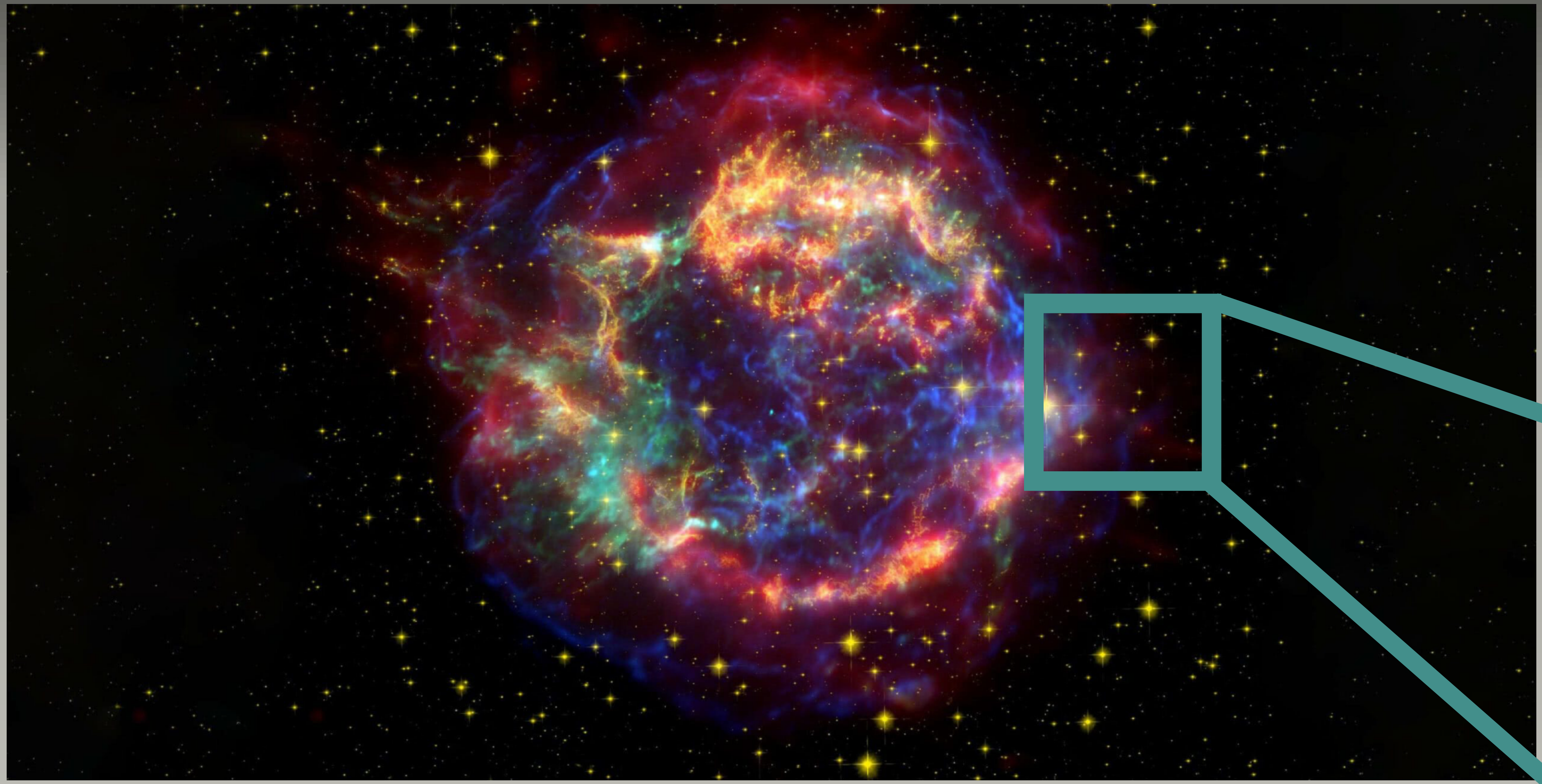

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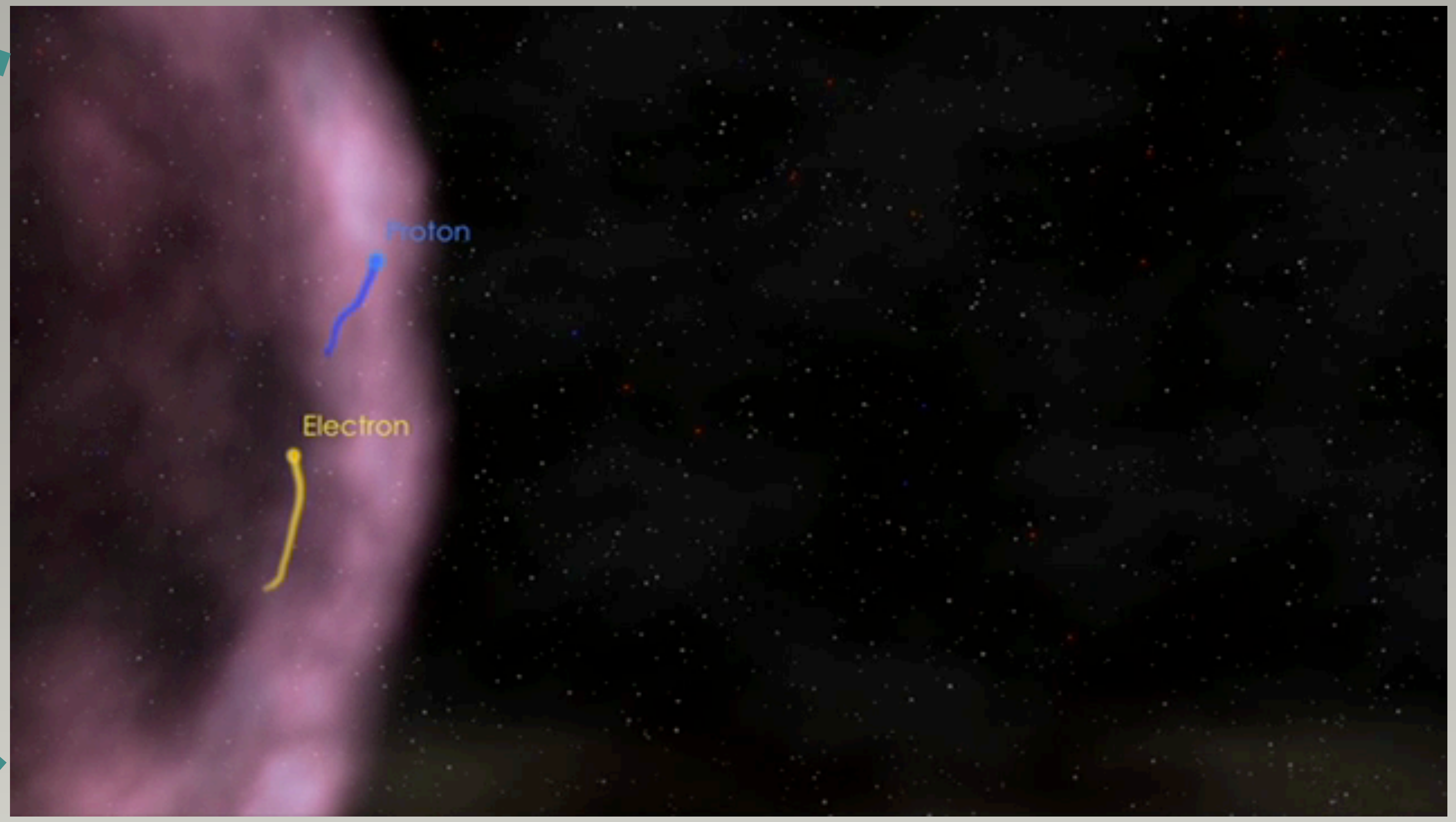

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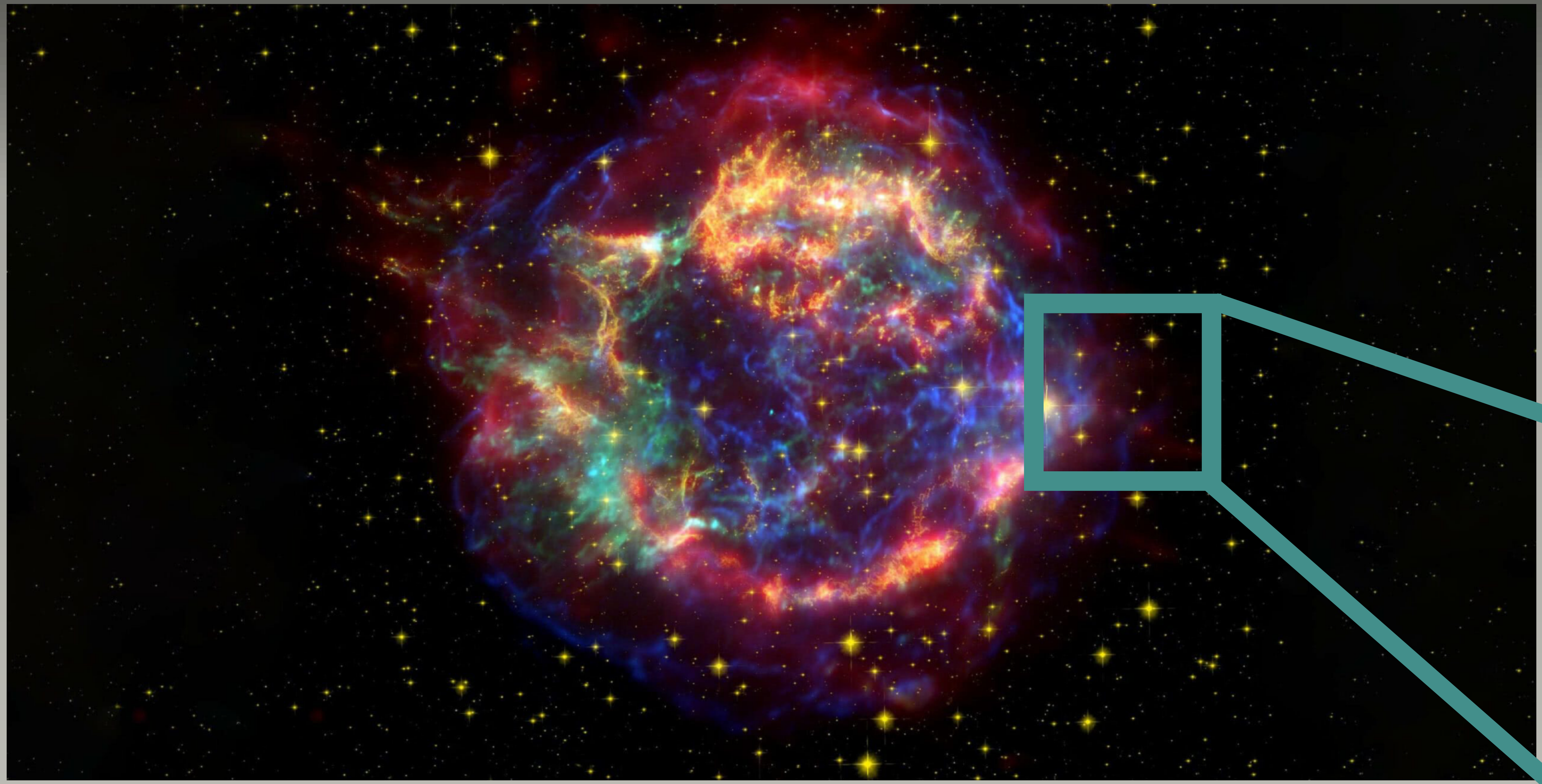

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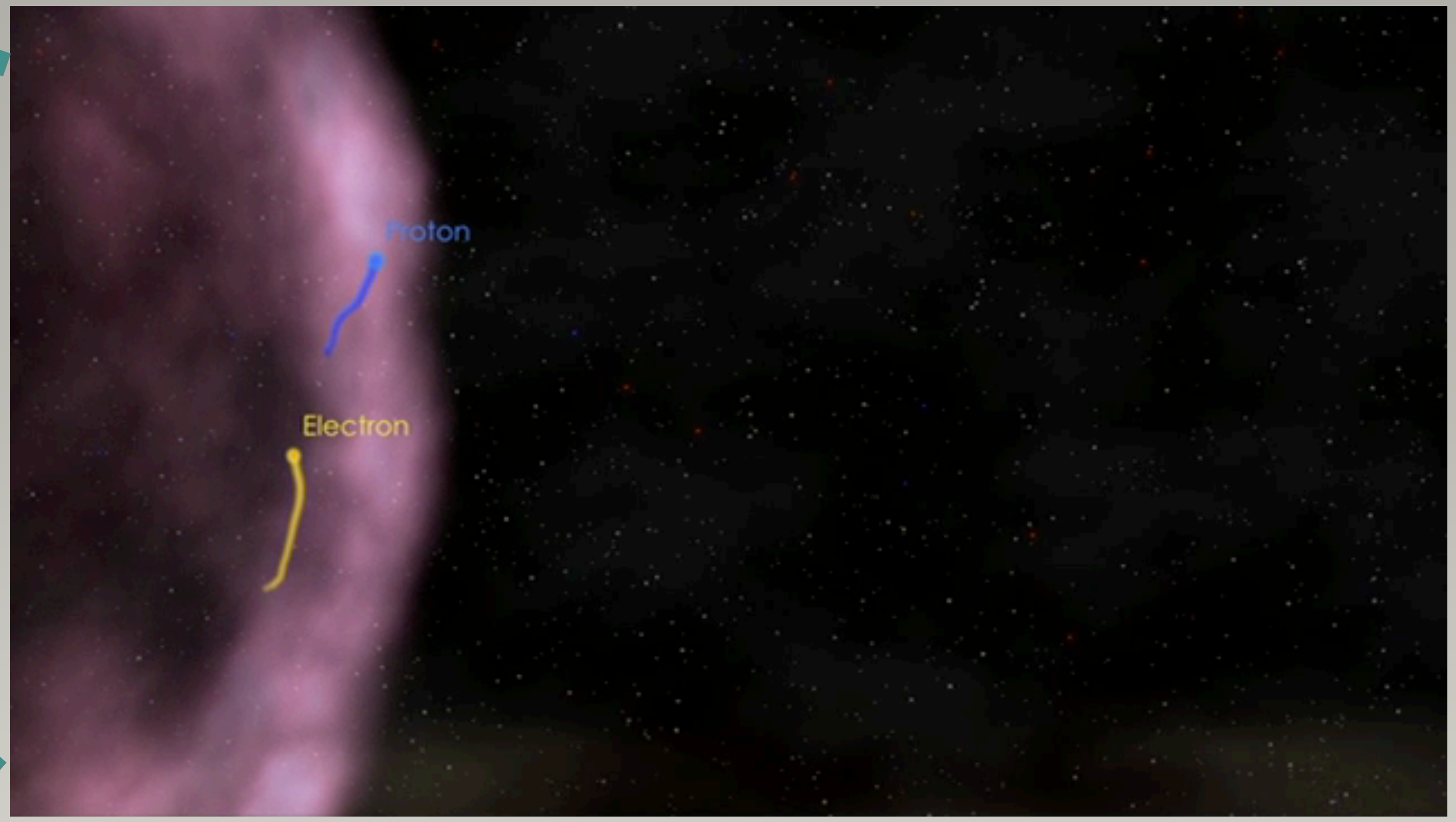

Assume particles are scattered frequently, keeping the distribution of particles close to isotropy
This assumption is usually accurate to order u/v (again we require v>> u)

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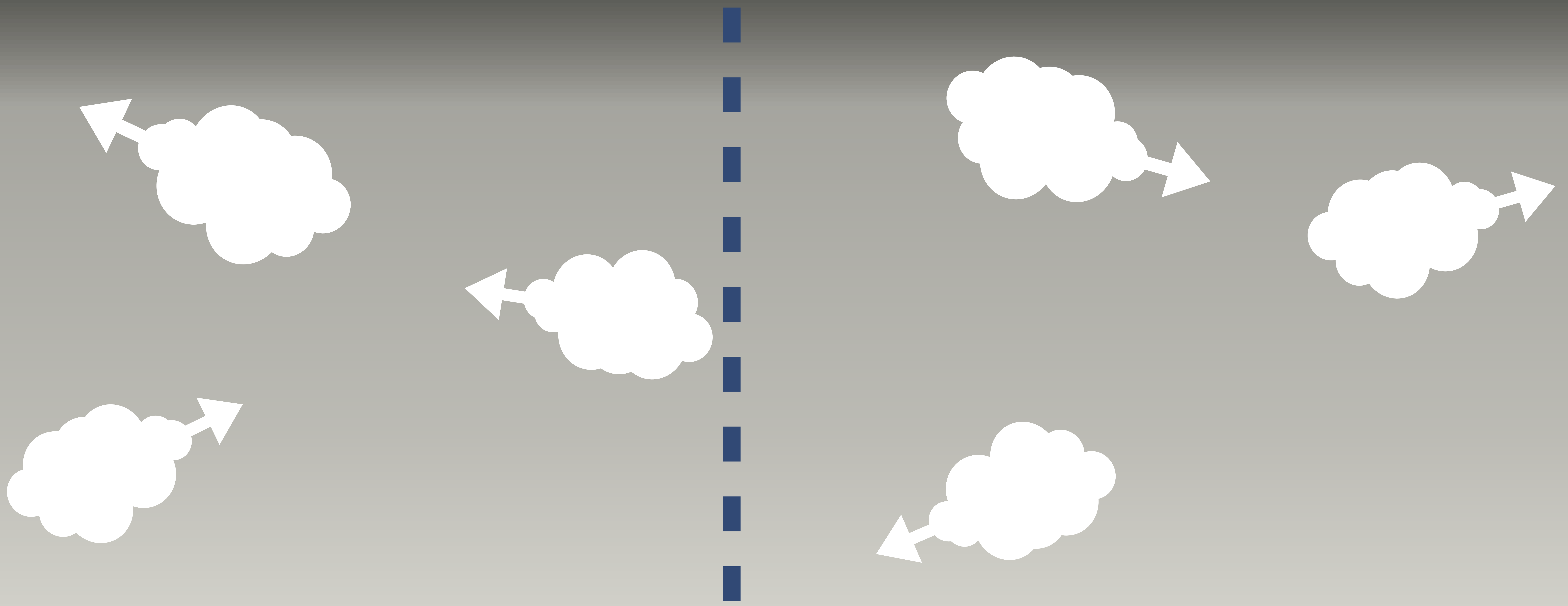
Acceleration at Shocks



Collisions are now (on average) head on.
Let's assume again that particle collisions are frequent, such that particle distribution is smooth about the shock and that the distribution stays close to isotropy

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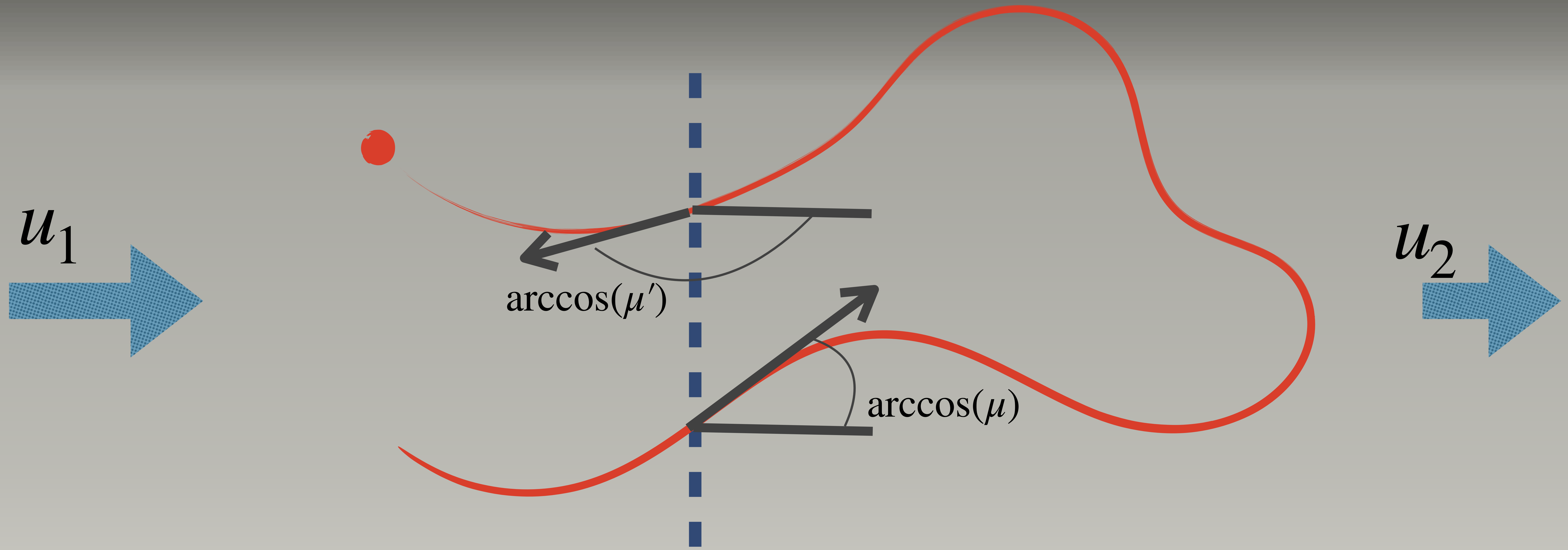
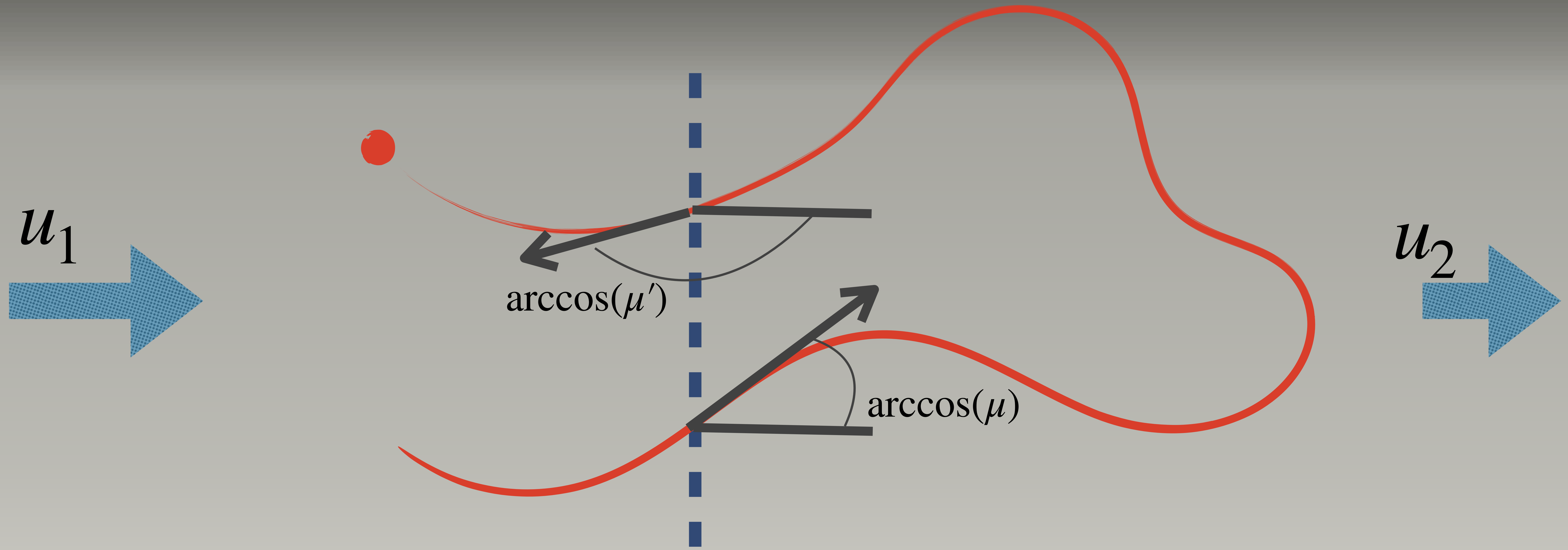
Average Energy/Momentum Change



One can show $\dfrac{\Delta p}{p} = \dfrac{\Delta u}{\mathrm{v}}(\mu - \mu')$ **(does this look familiar??)**

It is impractical to solve for all particles, so we need to average over the distribution. Luckily, here we can exploit the isotropy of the particles.

Average Energy/Momentum Change



If particles are isotropic in the local frame, the flux across the shock is proportional to $|v_x| = |v\mu|$

$$\frac{\langle \Delta p \rangle}{p} \approx \frac{\int_0^1 d\mu \ \int_{-1}^0 d\mu' |\mu\mu'| \frac{\Delta p}{p}}{\int_0^1 d\mu \ \int_{-1}^0 d\mu' |\mu\mu'|} = \frac{4\Delta u}{3v}$$

Note this is first order in (u/v)!

Producing power laws

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Recall the differential density $\int n(p)\; dp = \int d^3p\, f = \int_0^\infty dp p^2 \int_{-1}^{1} d\mu \int_0^{2\pi} d\phi\; f$

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This gives an escape probability $P_{esc} = \dfrac{\textbf{flux to DS infinity}}{\textbf{flux across shock}} = \dfrac{4u_2}{v}$ **(p indep.!!)**

Producing power laws

Applying the same approach as we did for Fermi's clouds

$$N_>(p + \langle \Delta p \rangle) = (1 - P_{\rm esc}) N_>(p), \quad \textbf{where recall } N_>(p) = \int_p^\infty n(p) dp$$

$$\frac{\partial N_>(p)}{\partial p} \langle \Delta p \rangle = - P_{esc} N_>(p) \quad \textbf{or} \quad \frac{\partial \ln N_>(p)}{\partial \ln p} = - \frac{P_{\rm esc}}{\langle \Delta p \rangle / p} = - \frac{3u_2}{u_1 - u_2} = - \frac{3}{r - 1}$$

Where $r = u_1/u_2$ is the compression ratio of the shock. Power-laws at last!!

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Where $r = u_1/u_2$ **is the compression ratio of the shock. Power-laws at last!!**

Note for strong (shock velocity >> sound/Alfven speed) $r \rightarrow 4$

This gives $N_>(p) \propto p^{-1}$**, or** $n(p) \propto p^{-2}$ **(c.f. the CR spectrum** $n_{cr}(p) \propto p^{-2.7}$**)**

Key Points

- Shocks are abrupt transitions of gas properties (density, velocity, Temperature, etc.)
- Are found in countless astrophysical sources, and are often associated to non-thermal emission
- Shocks naturally produce power-law particle spectrum (Why?)
- The energy gain is first order per cycle, and we call it a first order Fermi mechanism (or more generally diffusive shock acceleration or simply DSA for short)
- Key prediction $n \propto \frac{dN}{dE} \propto E^{-(r+2)/(r-1)}$ ($f \propto p^{-3r/(r-1)}$)
- power-law shape <u>independent of scattering</u>

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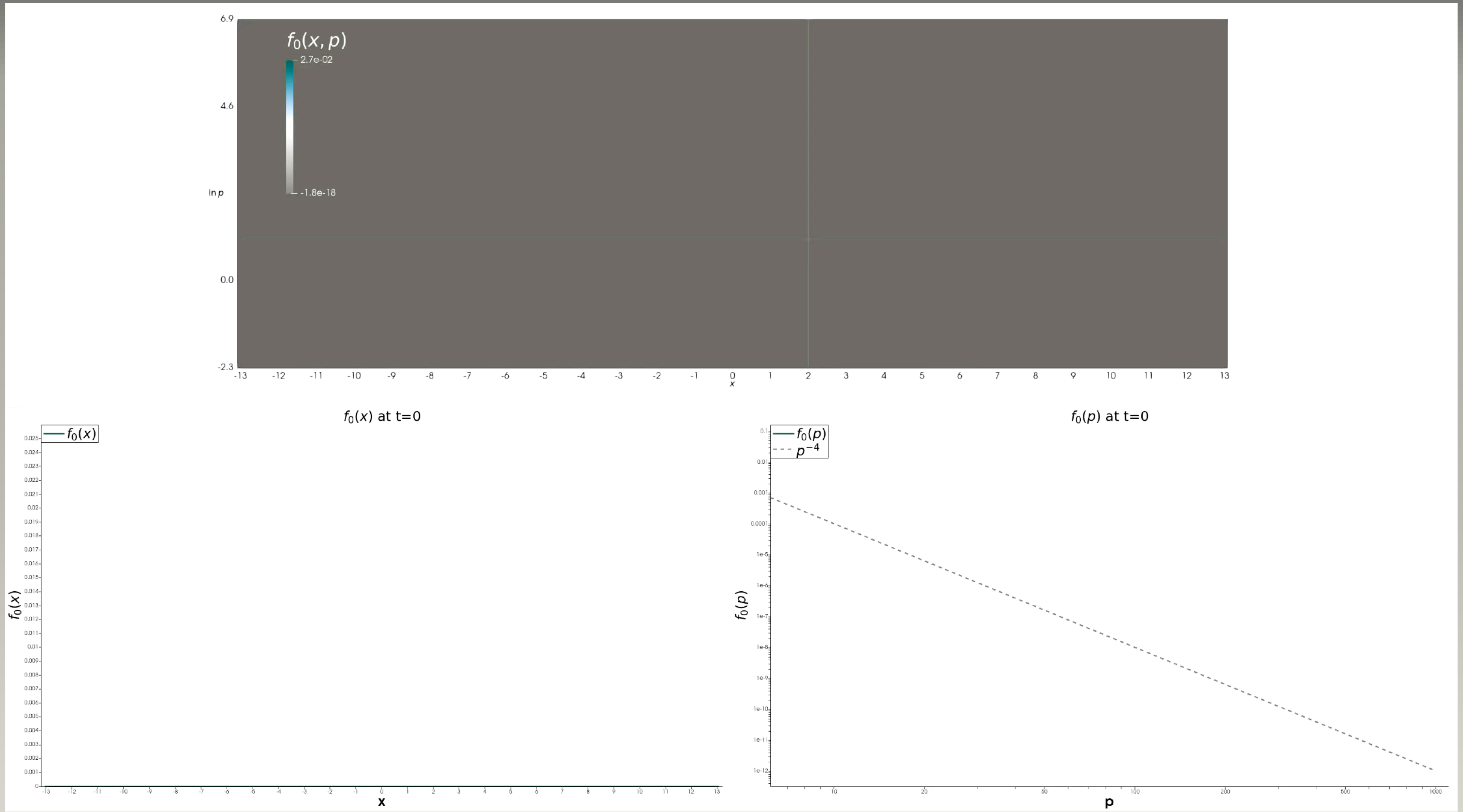
Diffusive Shock Acceleration in Action



Simulation of DSA by Nils Schween using SAPPHIRE code

Scattering is still critical
(It puts the D in DSA)

Upstream we see exponential decay
Balance of advection and diffusion

Downstream asymptotes to flat
Simply advective escape.

Maximum energy marches forward in time.

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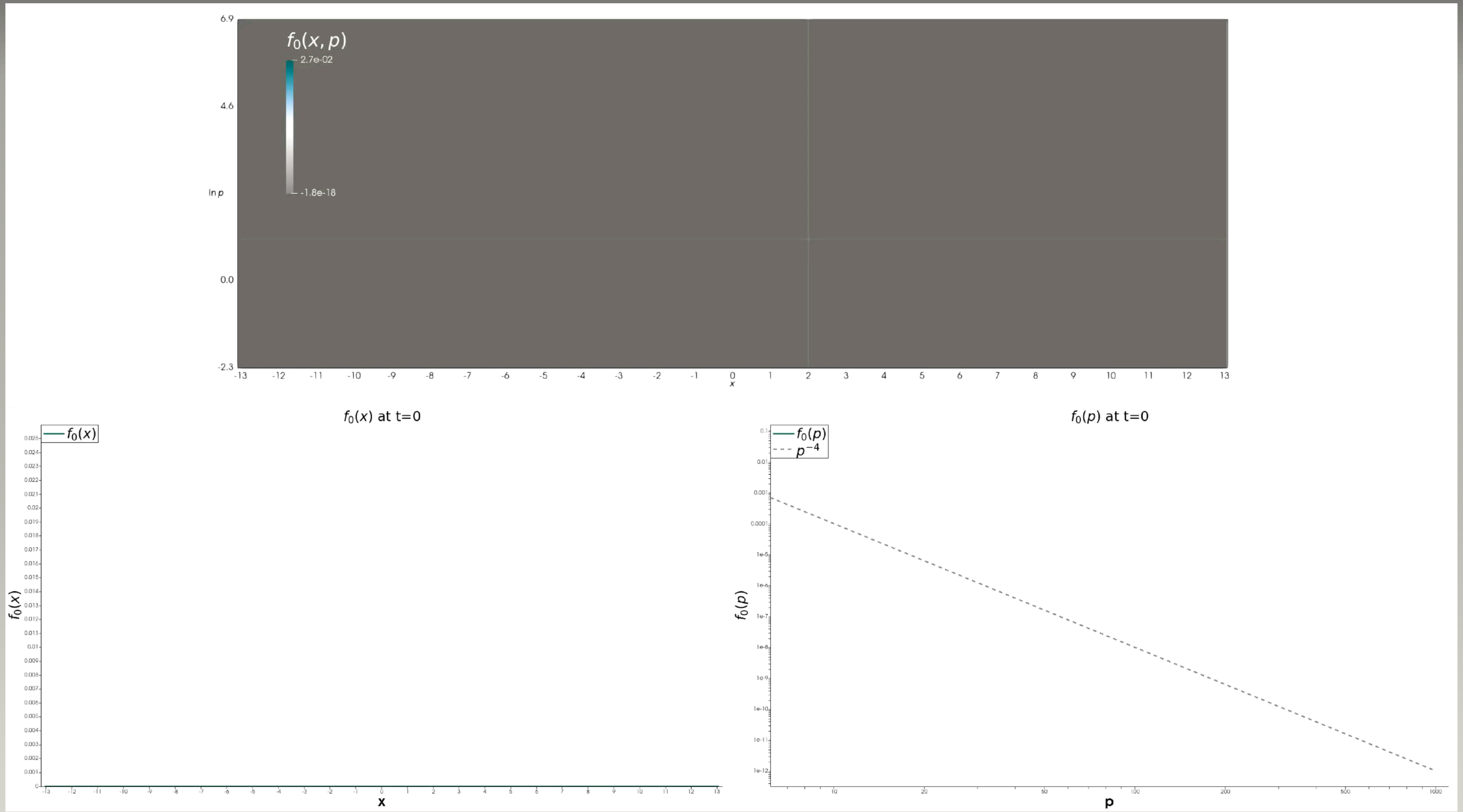

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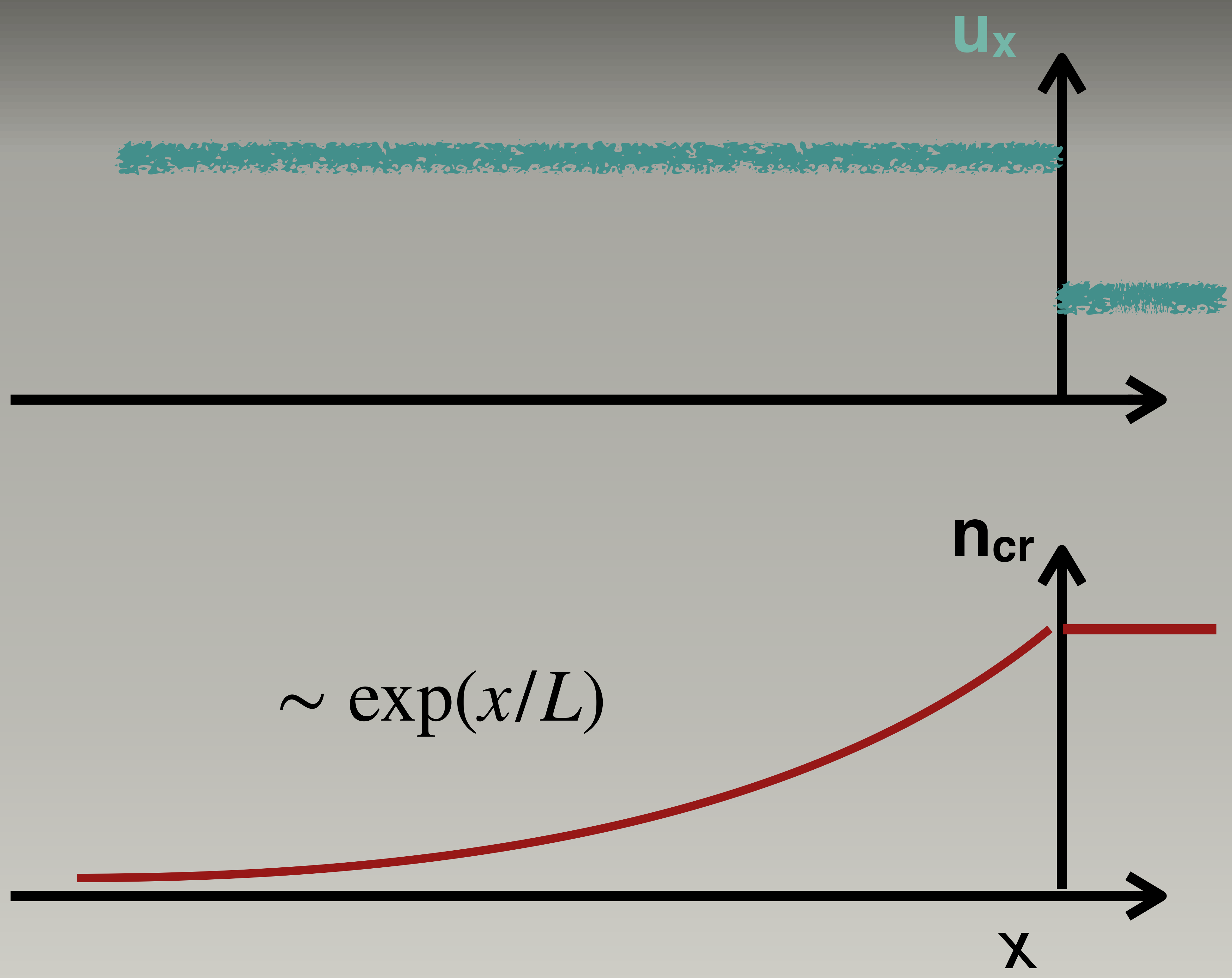
How Fast? How high?



We can describe everything in terms of a spatial diffusion coefficient D_{xx}

In upstream, steady state

$$u_1 \frac{\partial n}{\partial x} = \frac{\partial}{\partial x} D_{xx} \frac{\partial n}{\partial x} \quad \text{->} \quad n \propto e^{u_1 x / D_{xx}} \quad \mathbf{(L = D/u)}$$

Residence time upstream: $t_{us} \sim L/\mathrm{v} = D/u\mathrm{v}$

We define the acceleration time as $t_{\rm acc} = \frac{p}{\dot{p}} = \frac{t_{\rm cycle}}{\langle \Delta p \rangle / p} \approx \frac{D_{xx}}{u_1^2}$ **rate** $\propto (u/v)^2$

How Fast? How high?

We showed* $t_{\mathrm{acc}} \approx \dfrac{D_{xx}}{u_1^2}$ **. So what determines the maximum energy?**

*A more thorough derivation gives $t_{\mathrm{acc}} = \dfrac{3}{\Delta u}\left(\dfrac{D_1}{u_1} + \dfrac{D_2}{u_2}\right) \approx \dfrac{8D_{xx}}{u_1^2}$

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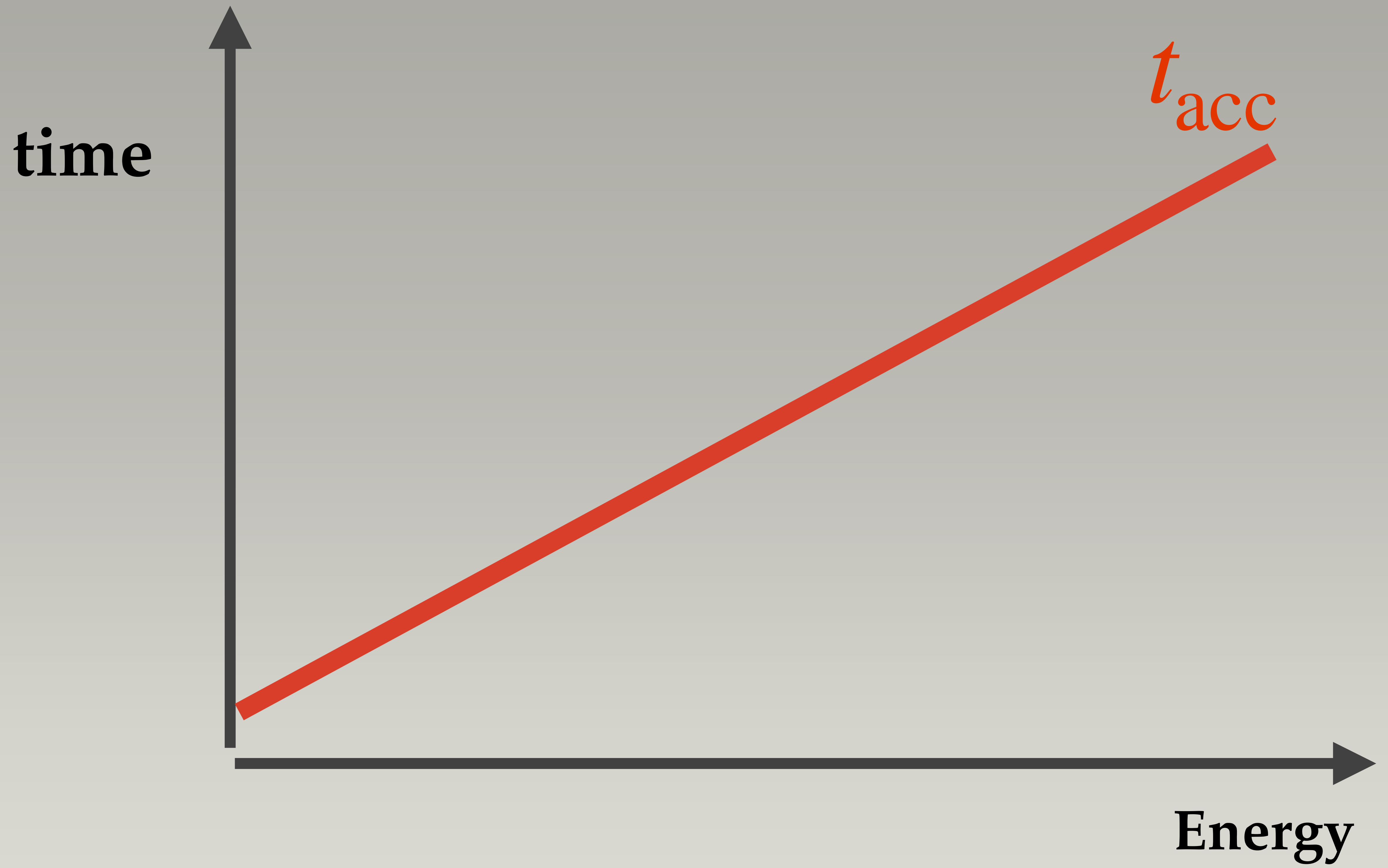

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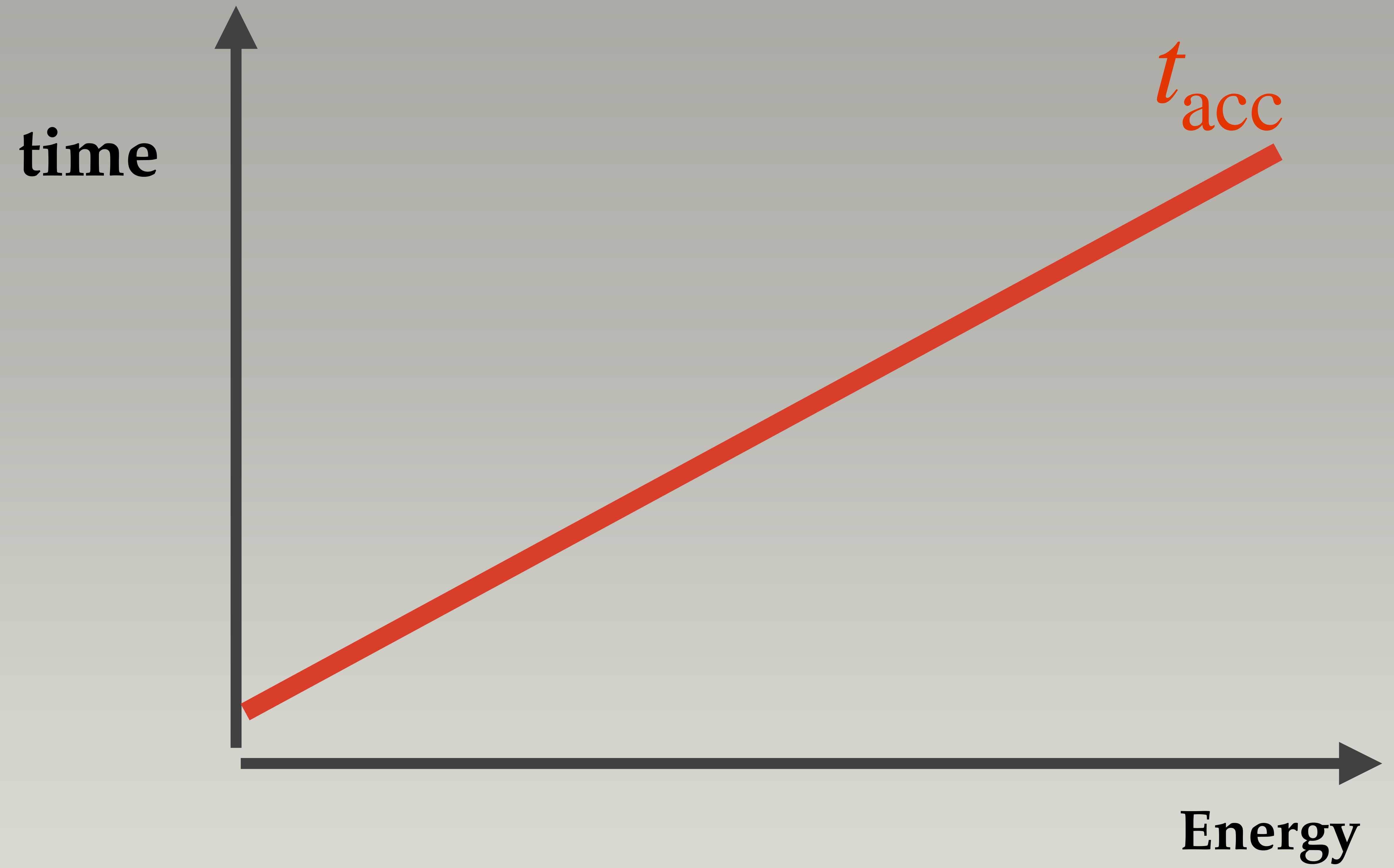

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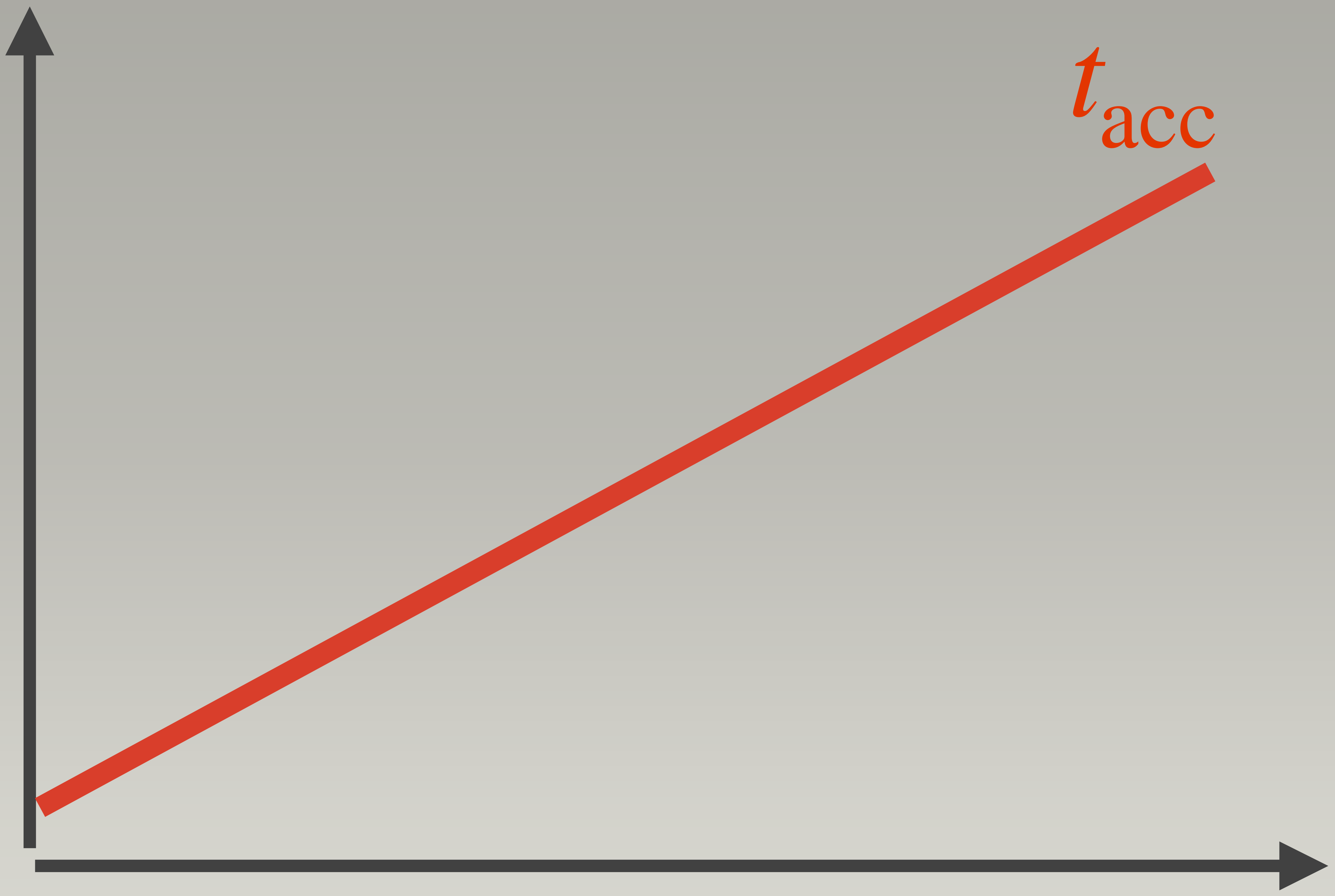

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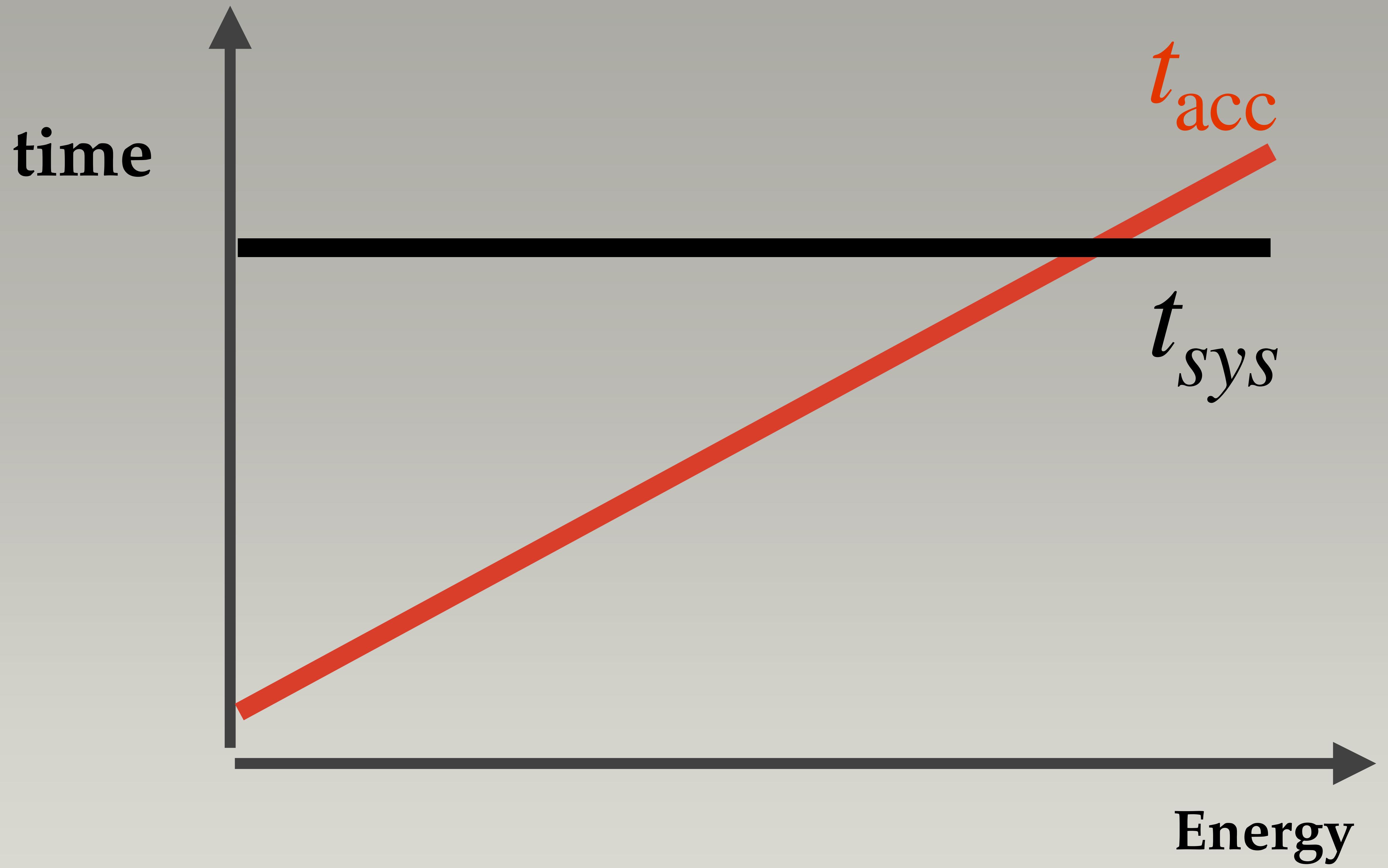

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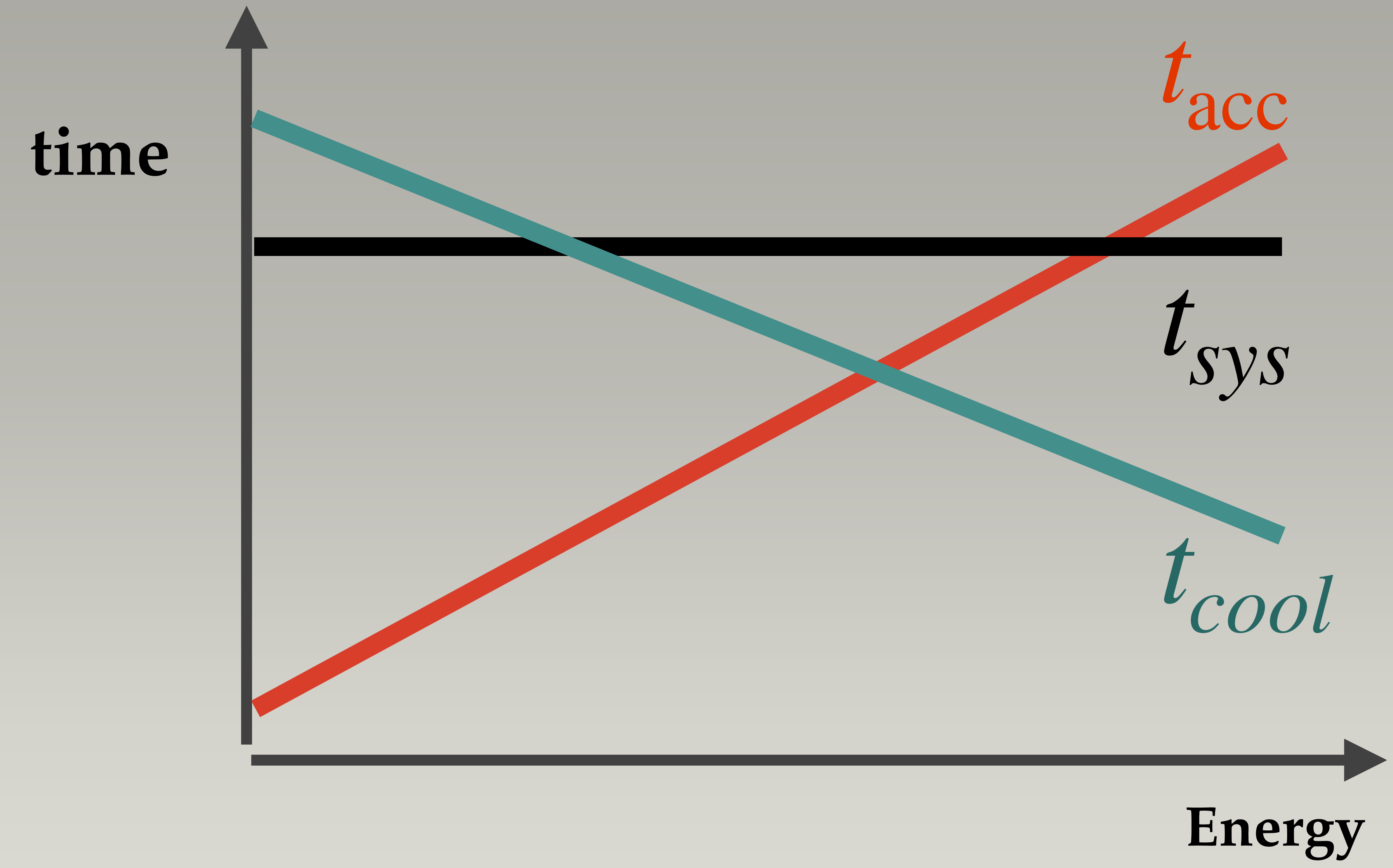

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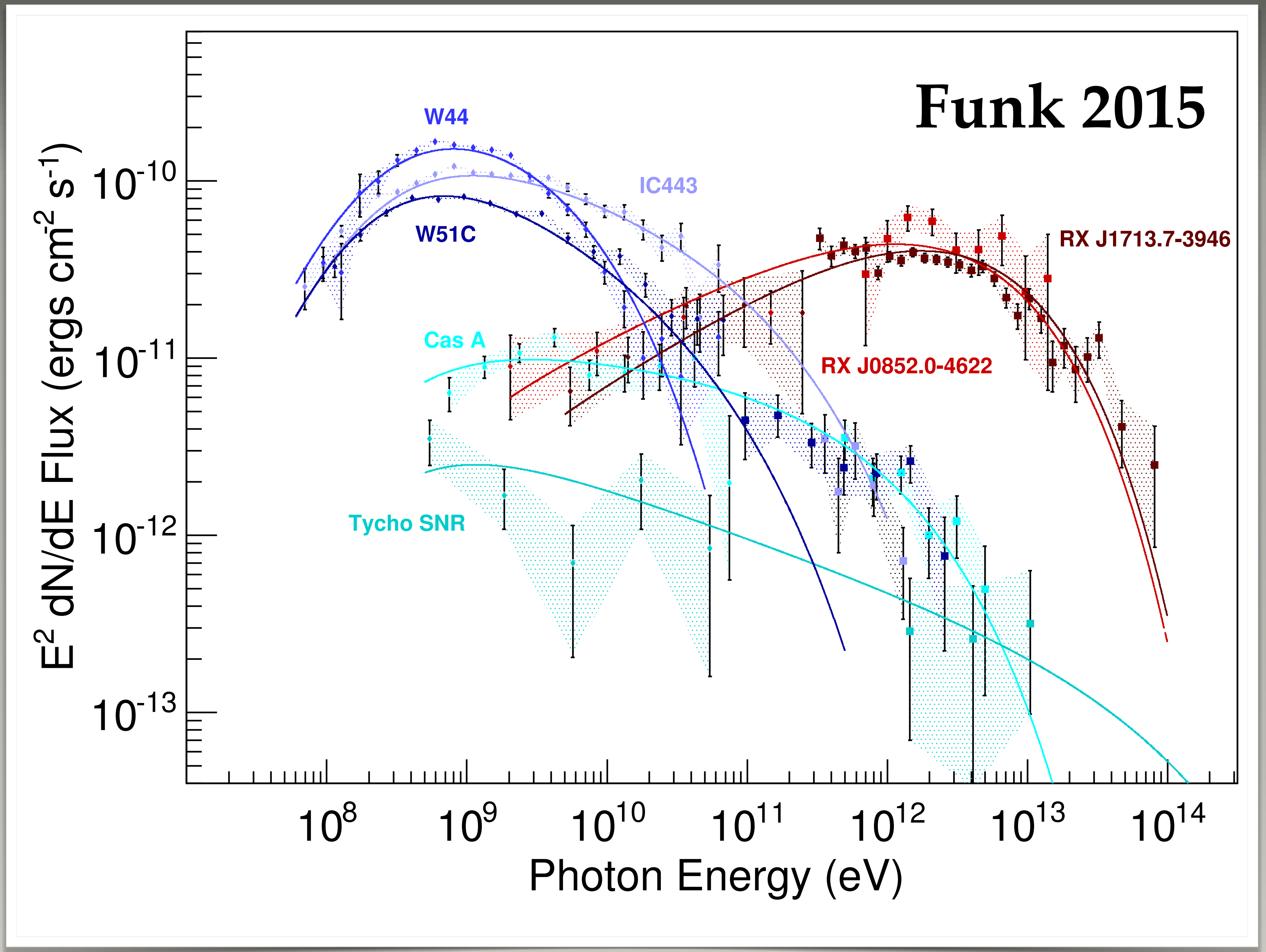
But DOES IT?

CR connection - Evidence/Contradiction



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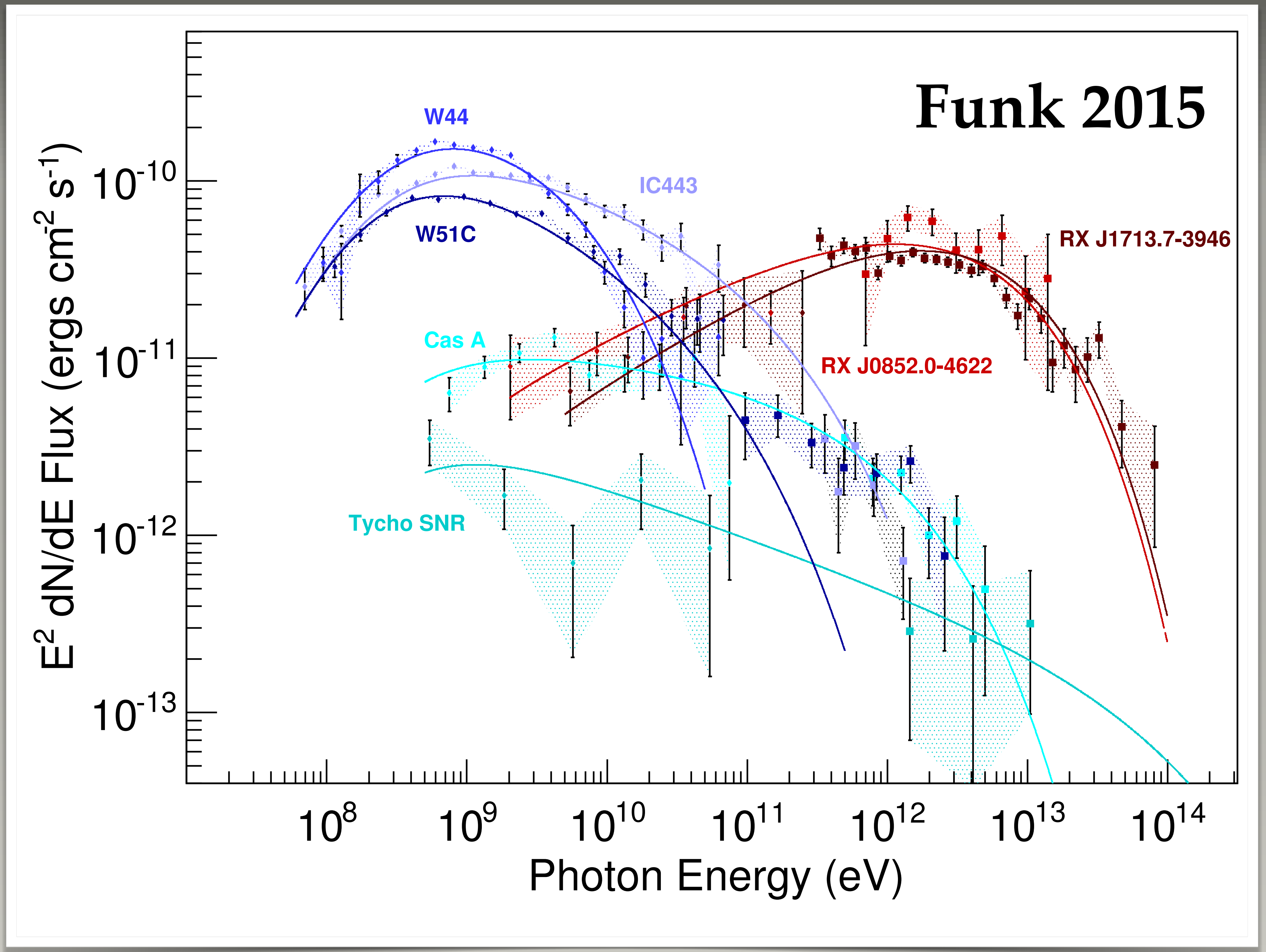

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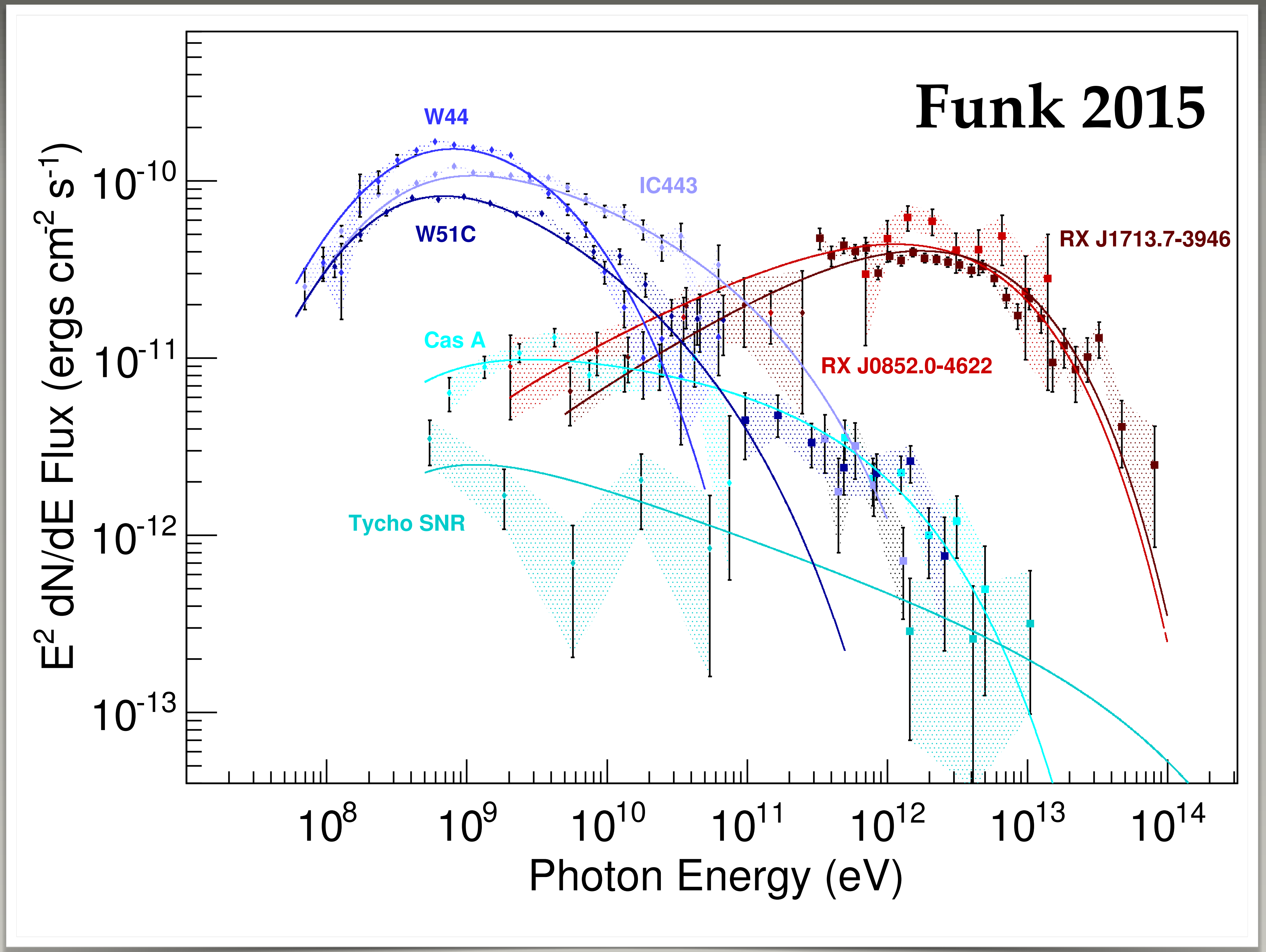

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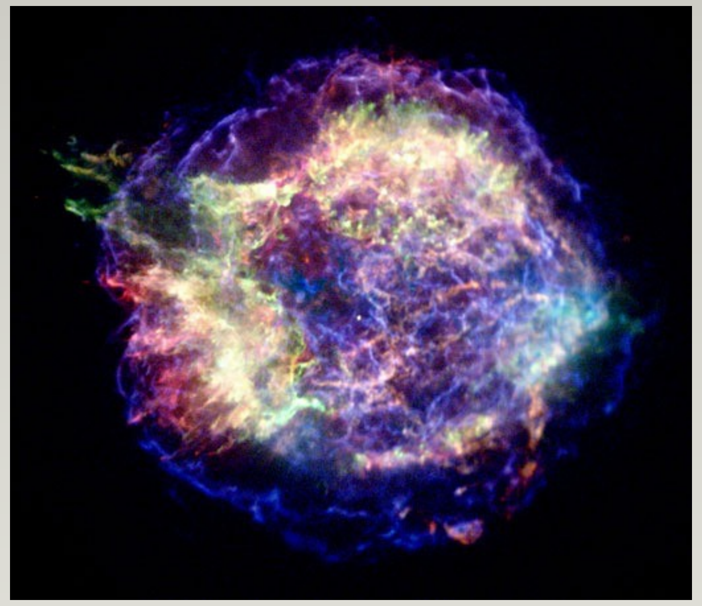
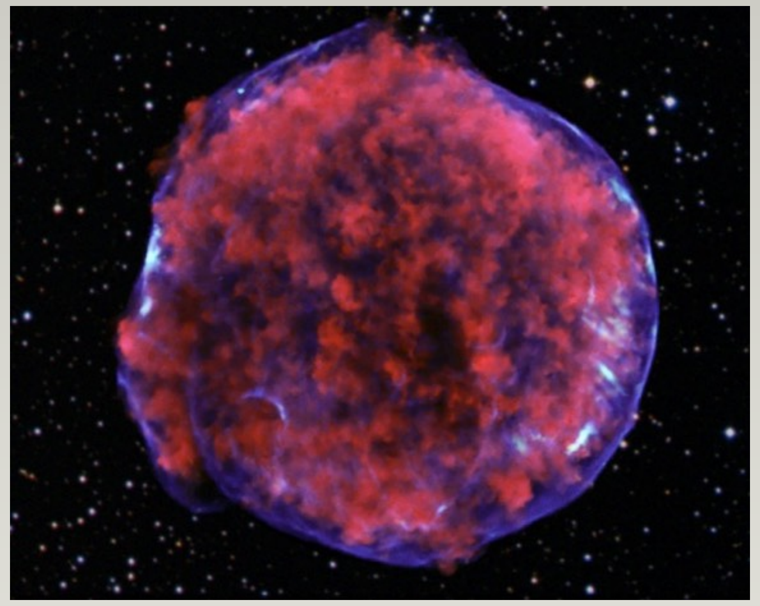

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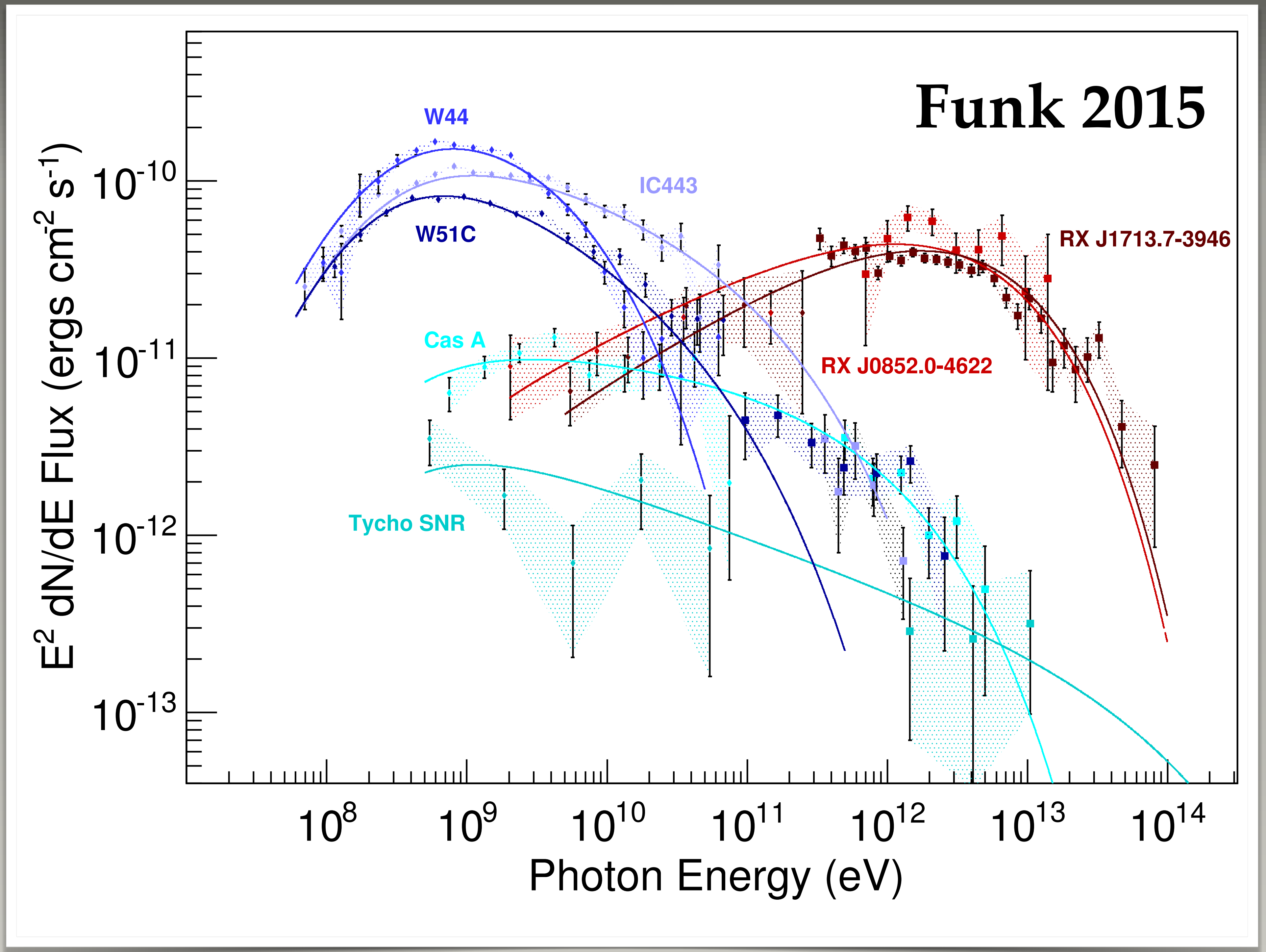

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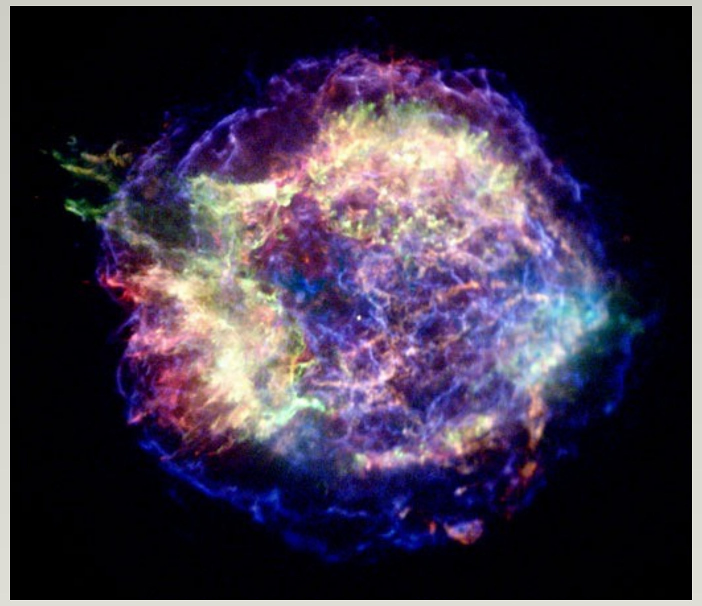

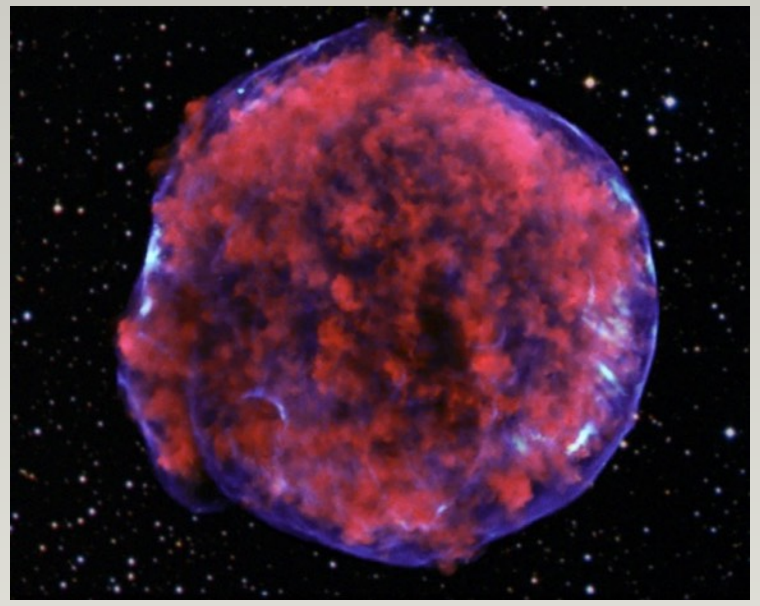

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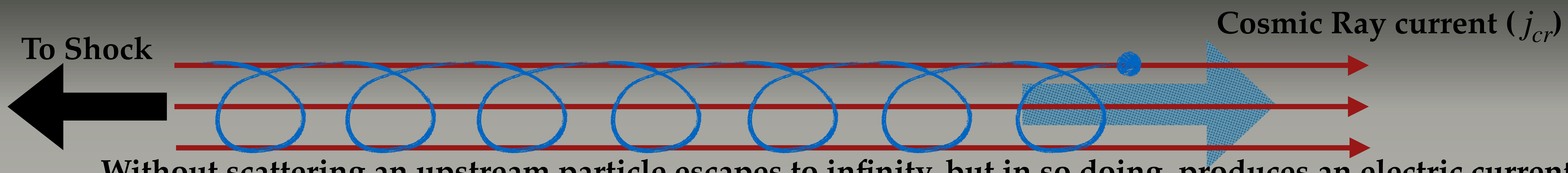
Do we even expect Bohm diffusion?



Without scattering an upstream particle escapes to infinity, but in so doing, produces an electric current

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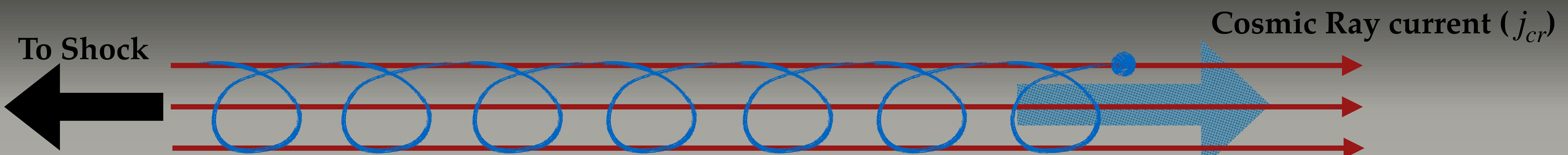

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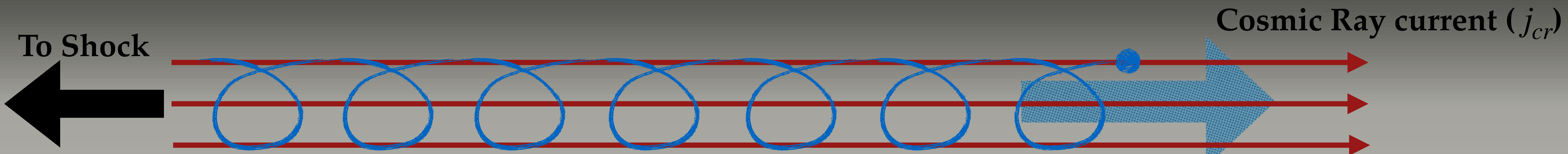

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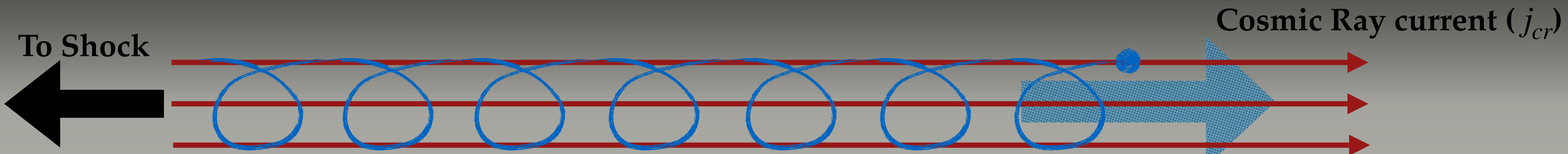

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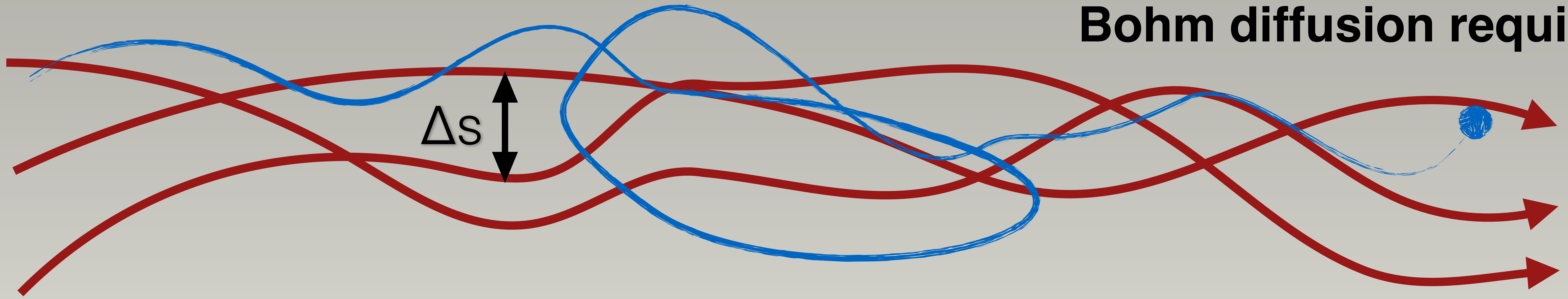

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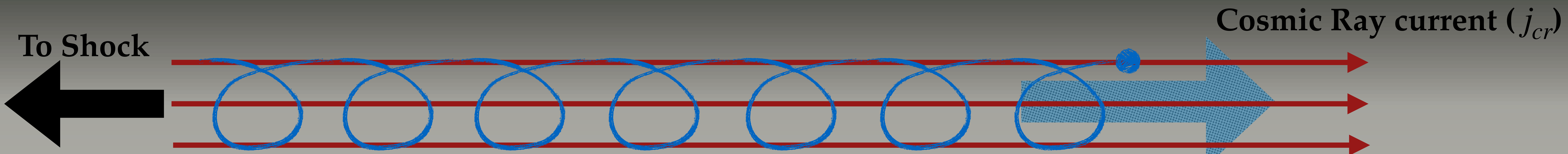

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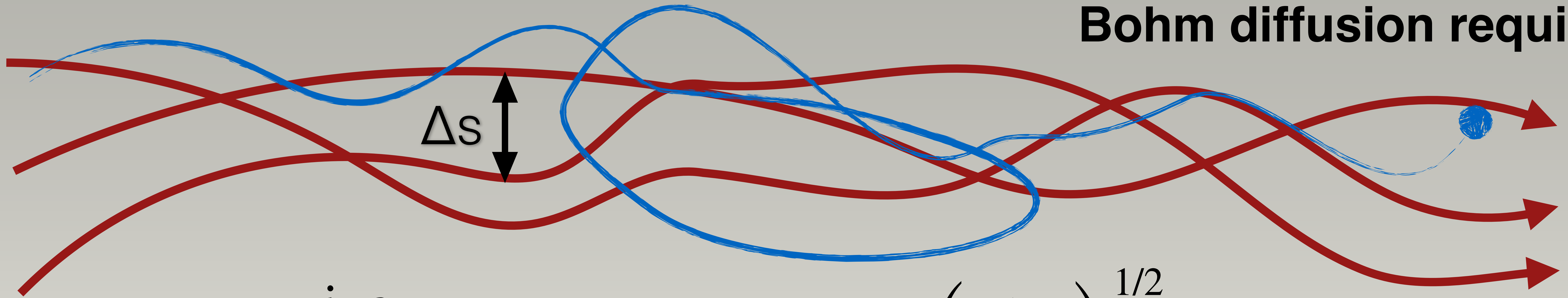

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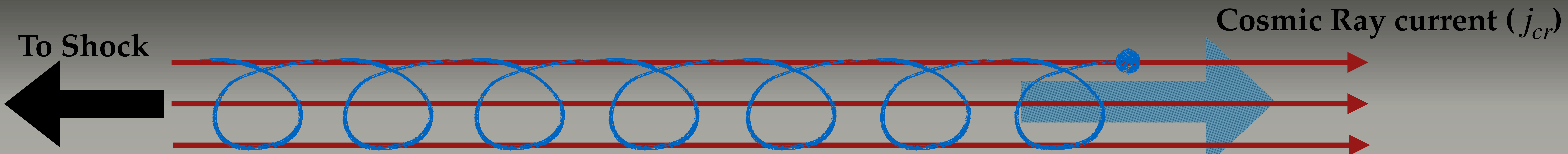

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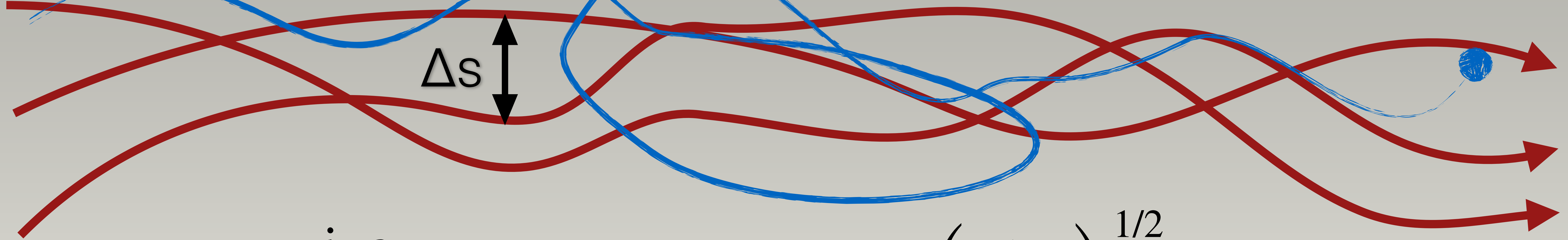

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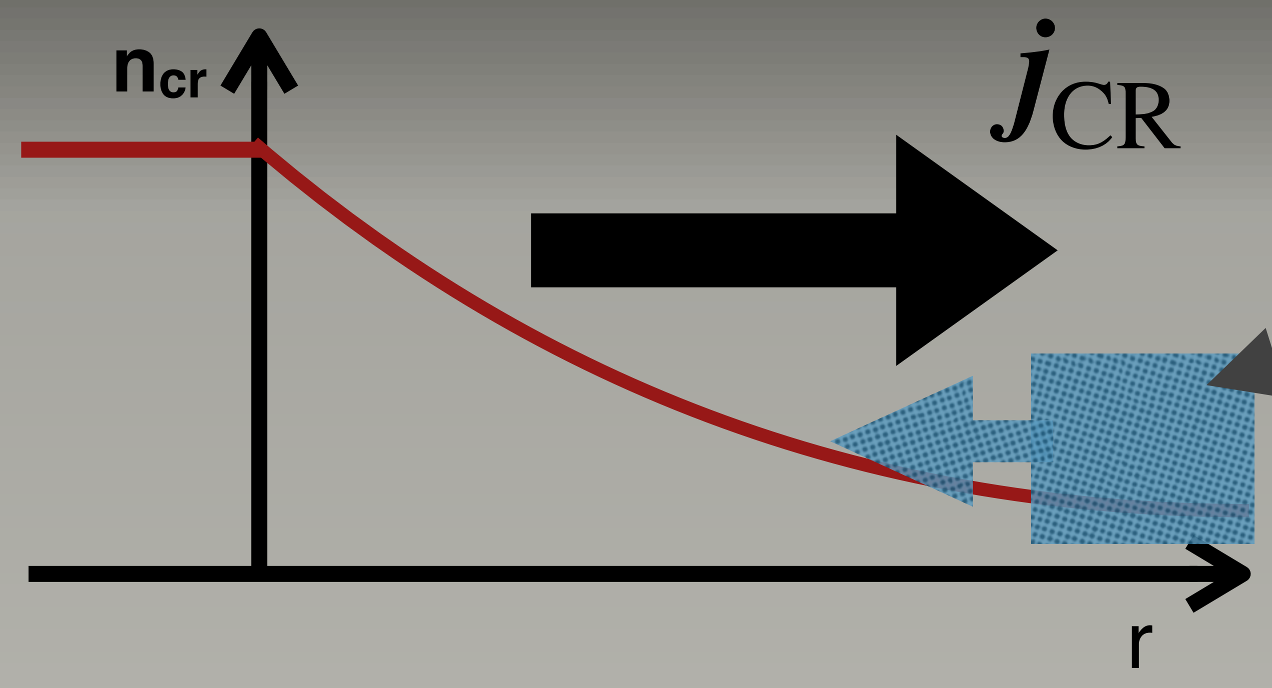
Consistent with Hillas/Lagage Cesarksy ($\eta \sim u_{sh}/c$). Doesn't solve PeV problem!!!

How to gain from magnetic field amplification?



MHD
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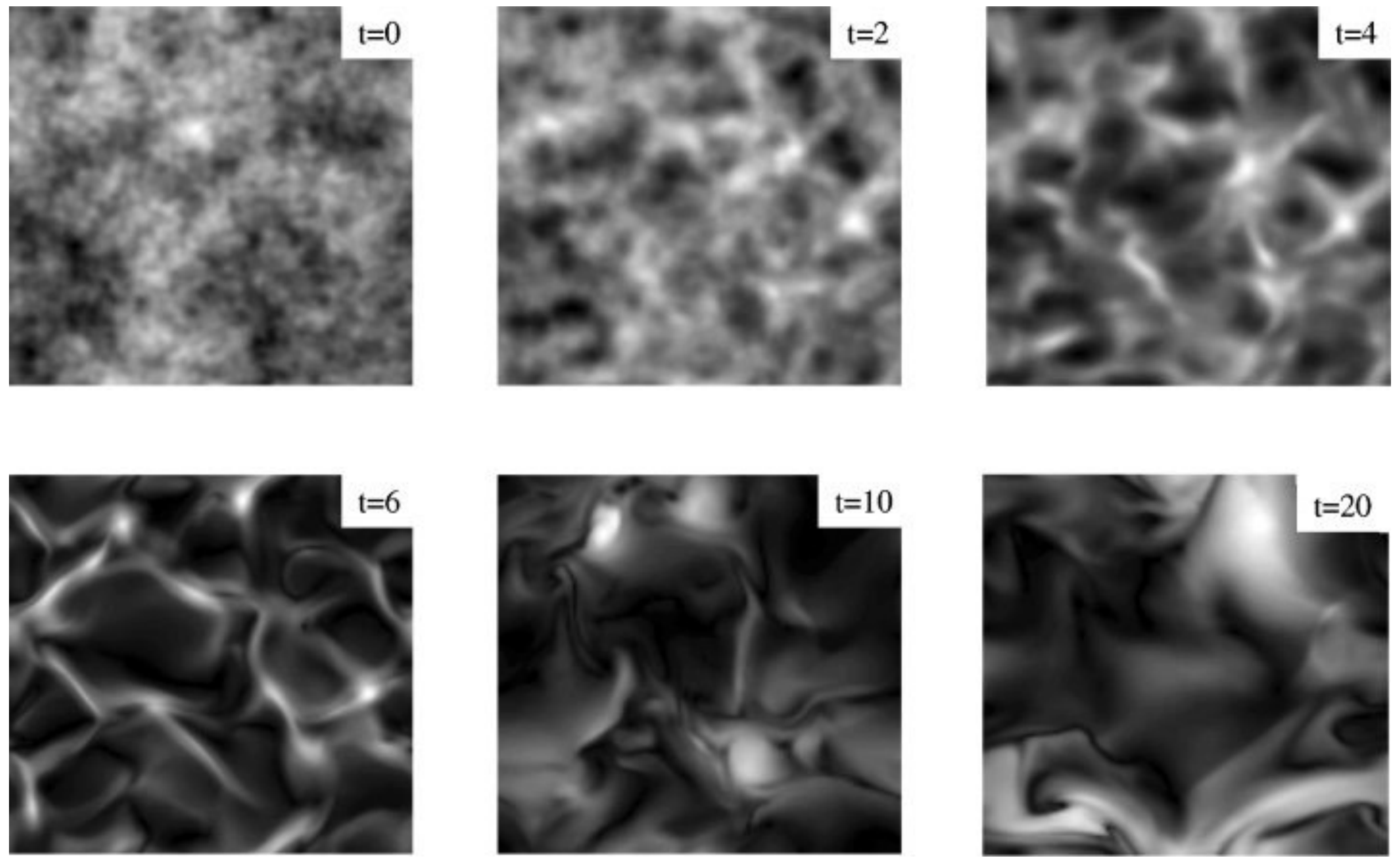

Bell 2004

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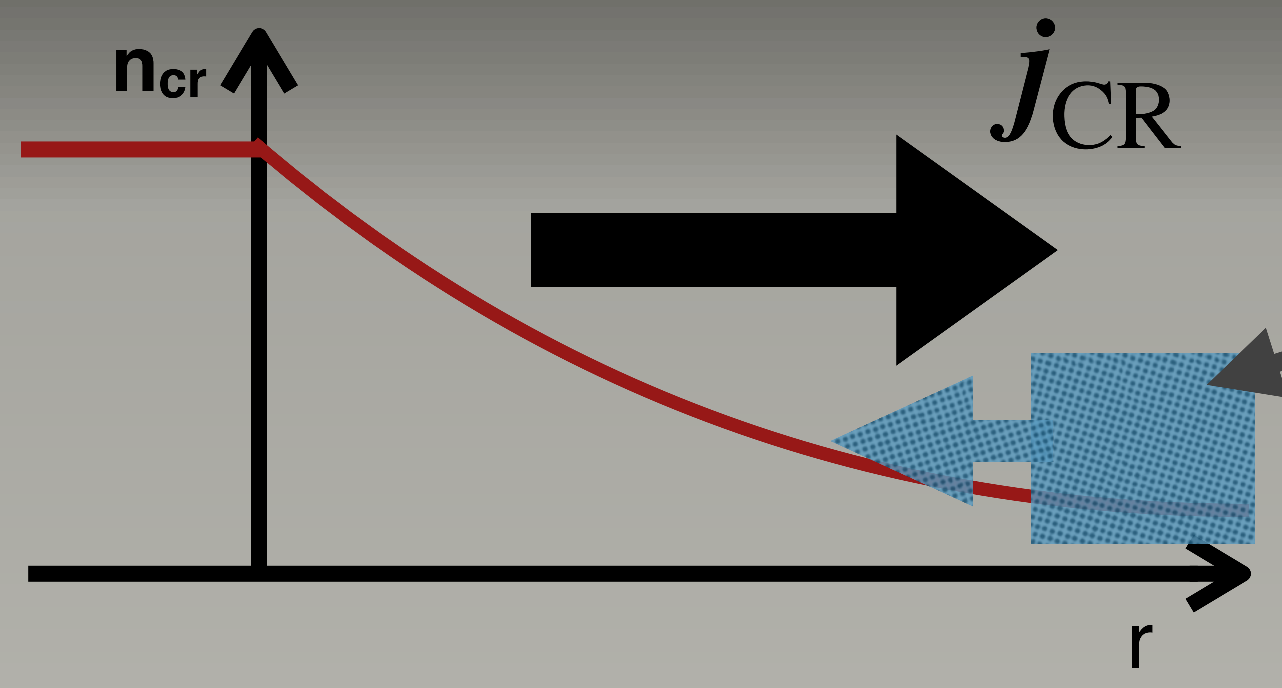

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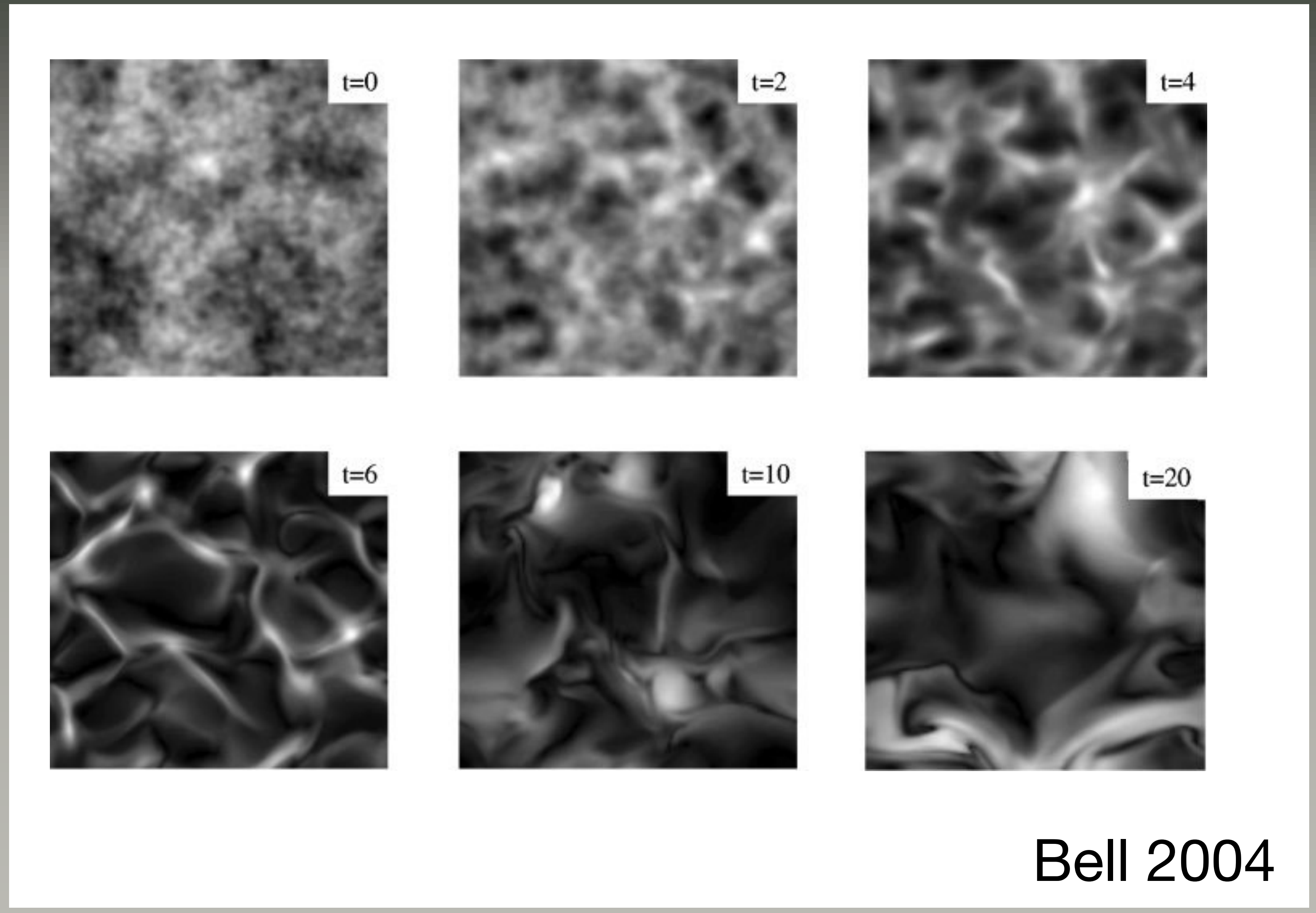

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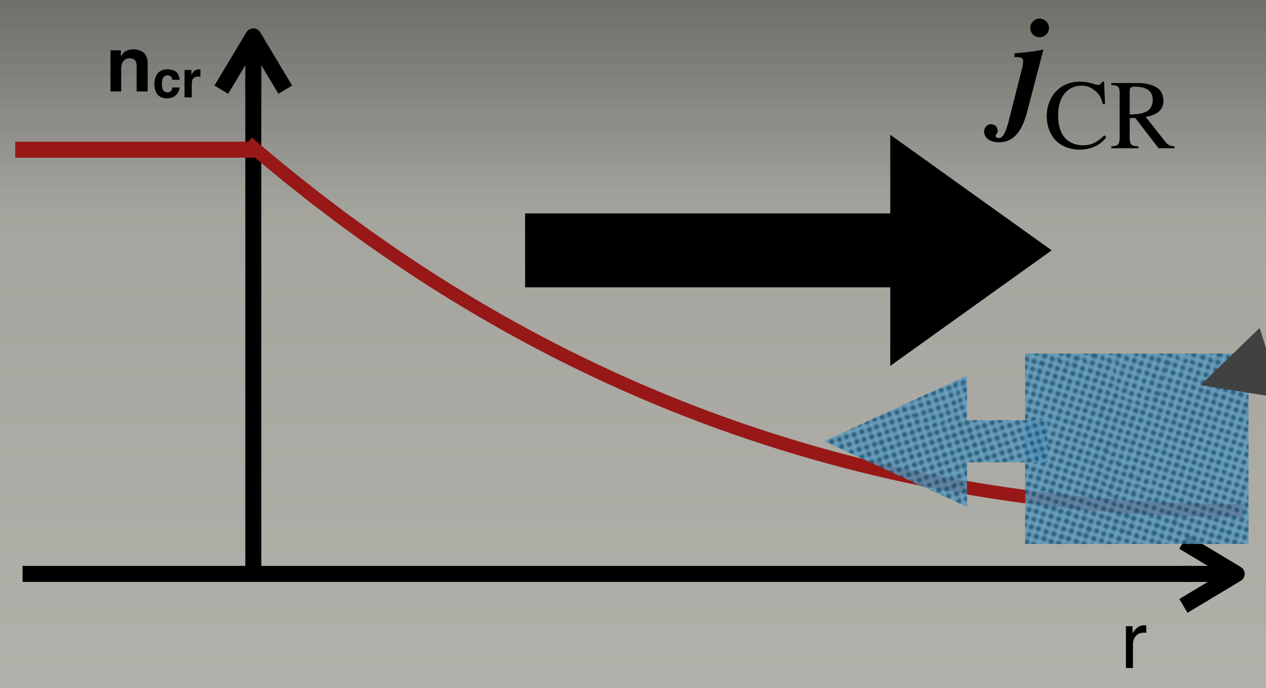

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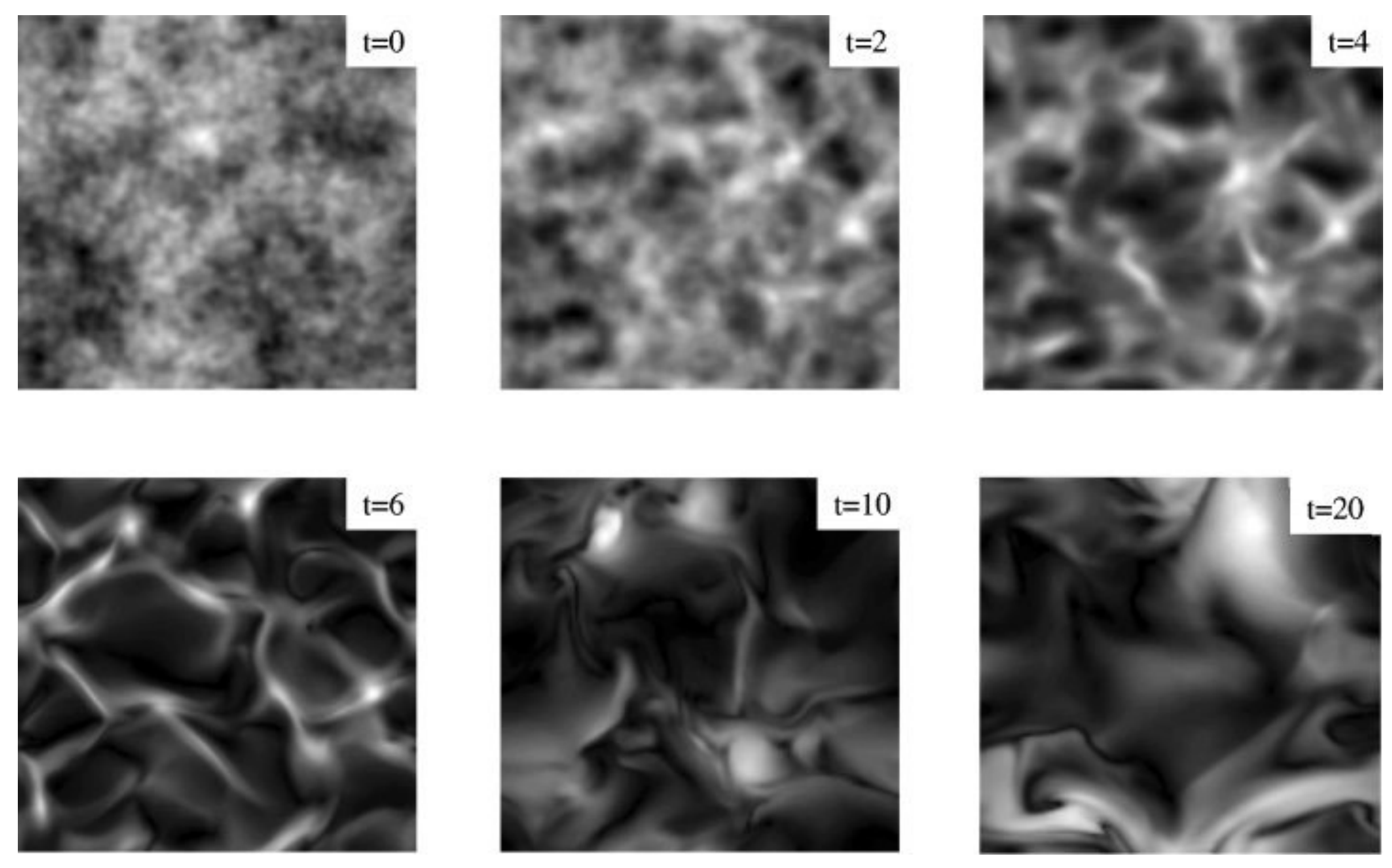

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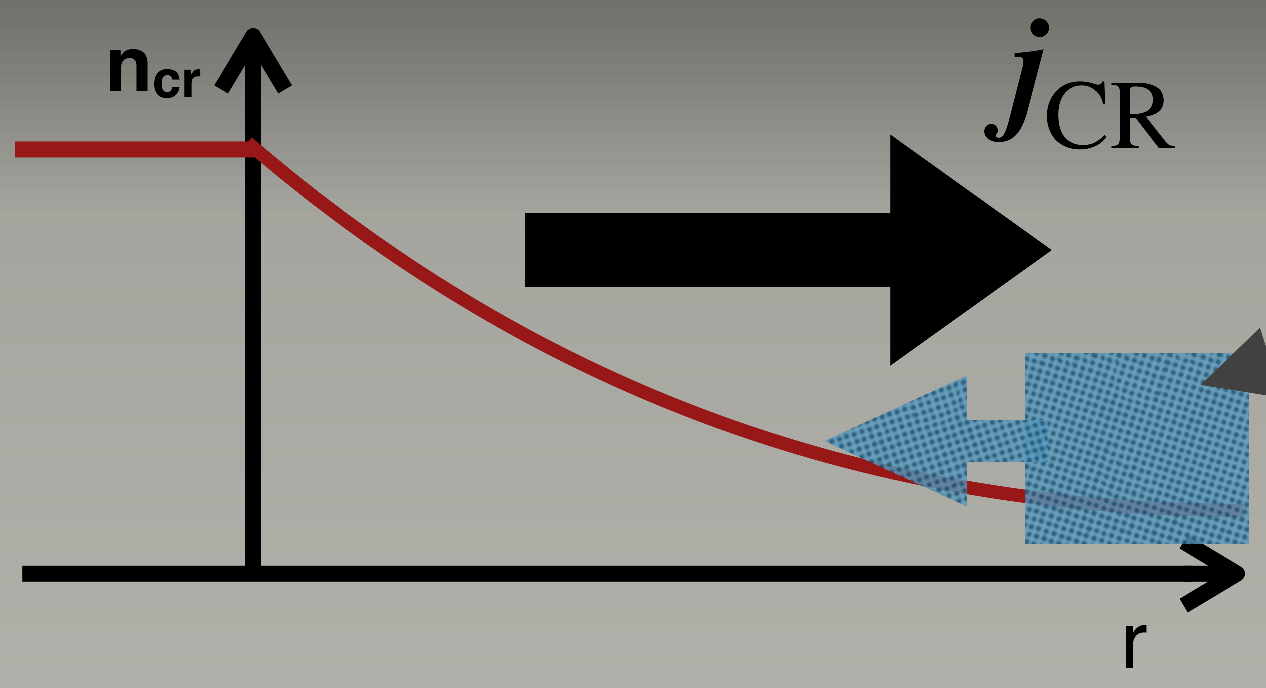

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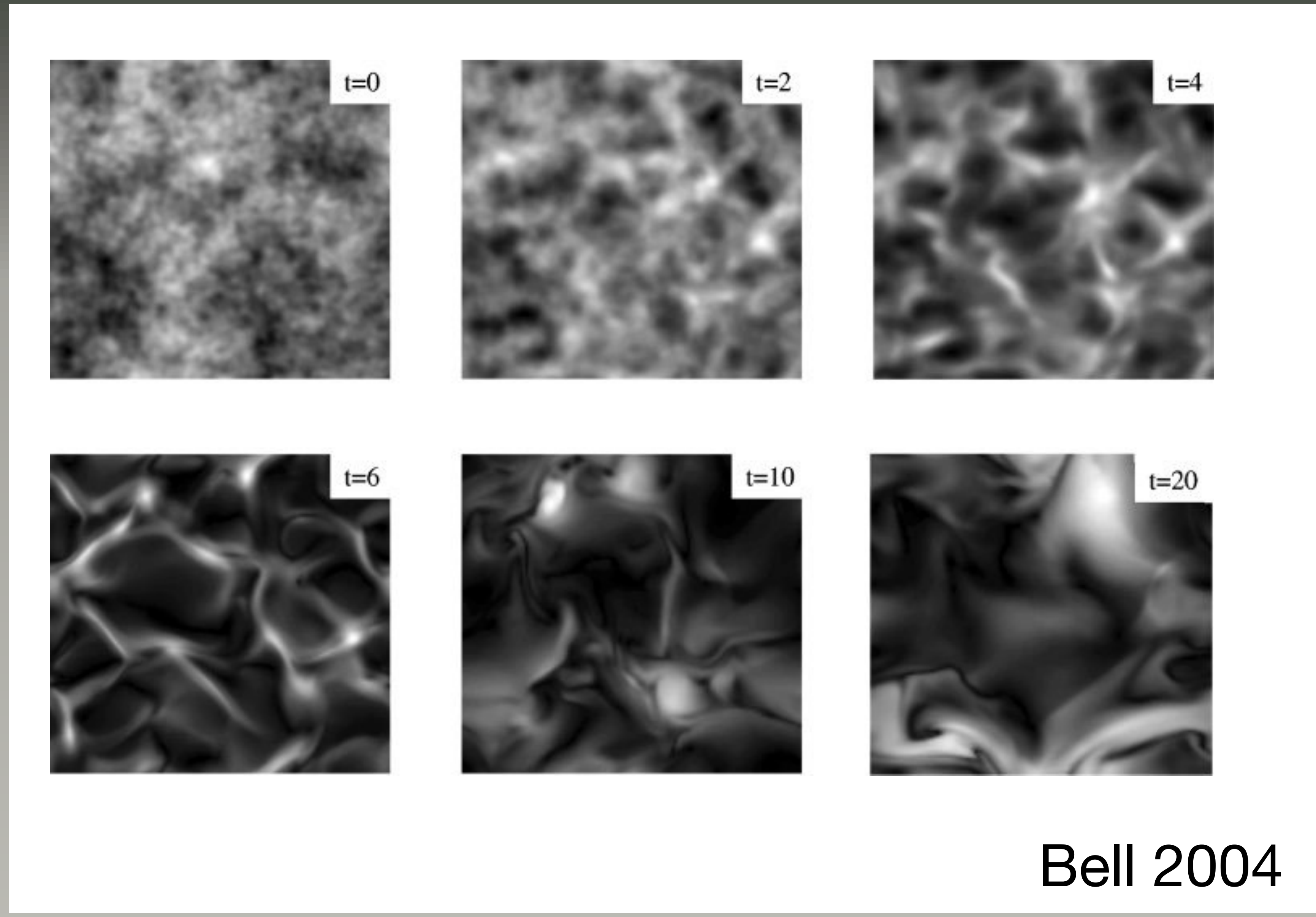


On scales < CR gyroradius, CRs are rigid
Current drives growth most rapidly on SMALL scales

When energy in amplified magnetic field exceeds that in background, field grows to larger scales. We set a (crude) confinement condition for 5-10 instability growth times.

$$\varepsilon_{\max} \approx 100\sqrt{n}\left(\frac{P_{\rm cr}}{\rho u_{\rm sh}^2}\right)\left(\frac{u_{\rm sh}}{5{,}000\ {\rm km\ s^{-1}}}\right)^3\left(\frac{t_{\rm snr}}{100\ {\rm yrs}}\right)\ {\rm TeV}$$

In principle possible to accelerate to PeV in young, fast SNRs in dense environments (winds?)

Key Points

- The acceleration time is $t_{\rm acc} \approx \dfrac{D_{xx}}{u_1^2} \propto (u/v)^{-2}$

- The maximum energy from DSA is close to Hillas limit if scattering is close to the magnetised limit, the so-called Bohm limit.

- Scattering is mediated by self-generated fluctuations (by escaping CRs)

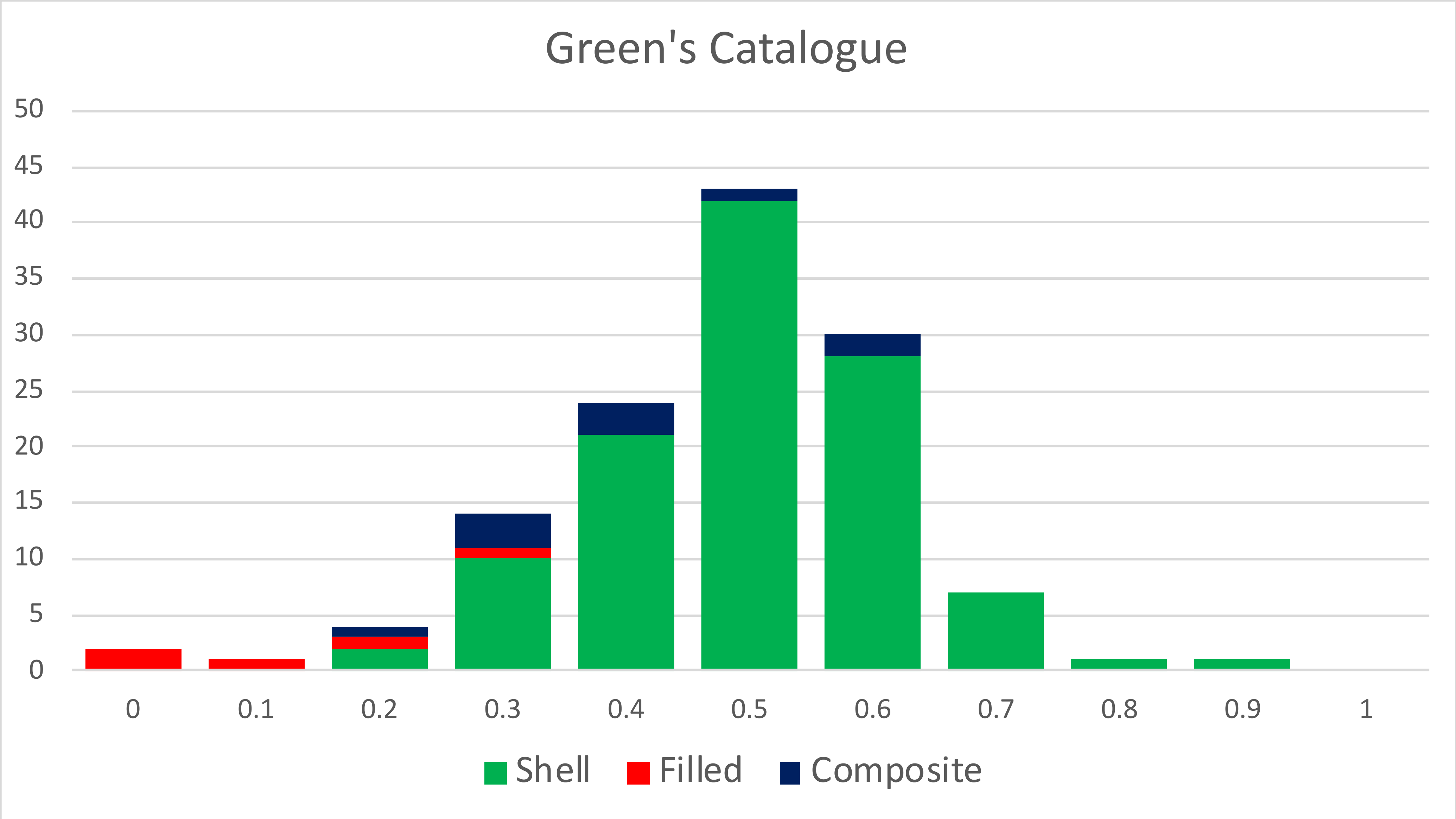
- To accelerate to PeV energies, requires strong magnetic field amplification. Confinement appears to be the limiting factor.

- Requires acceleration to be EFFICIENT.

Yet another problem…..

A power-law of electrons produces a power-law synchrotron flux (tomorrow's lecture)

If $dN/dE \propto E^{-\gamma}$, $F_\nu \propto \nu^{-\alpha}$ **where** $\alpha = \dfrac{\gamma - 1}{2}$



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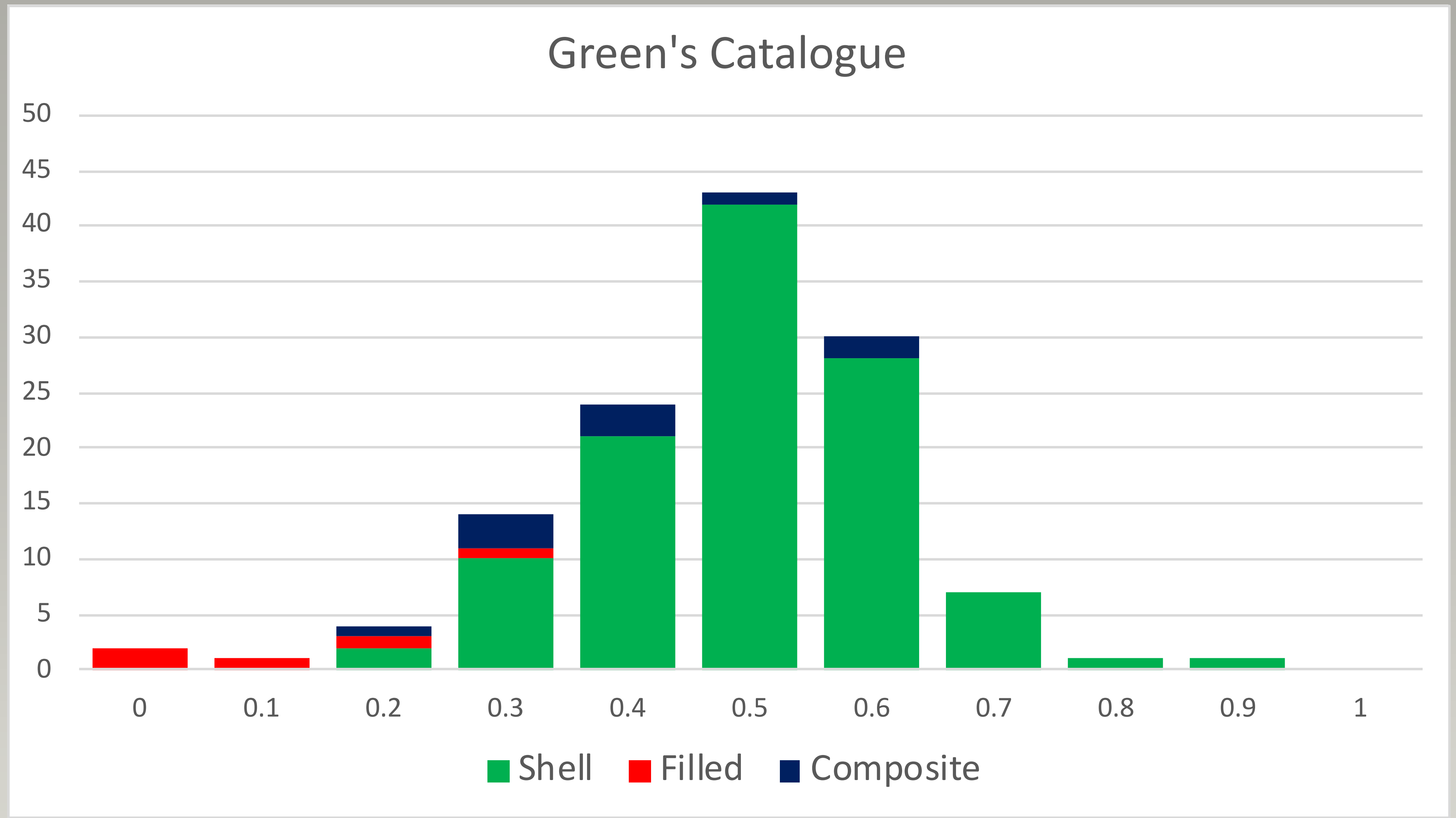
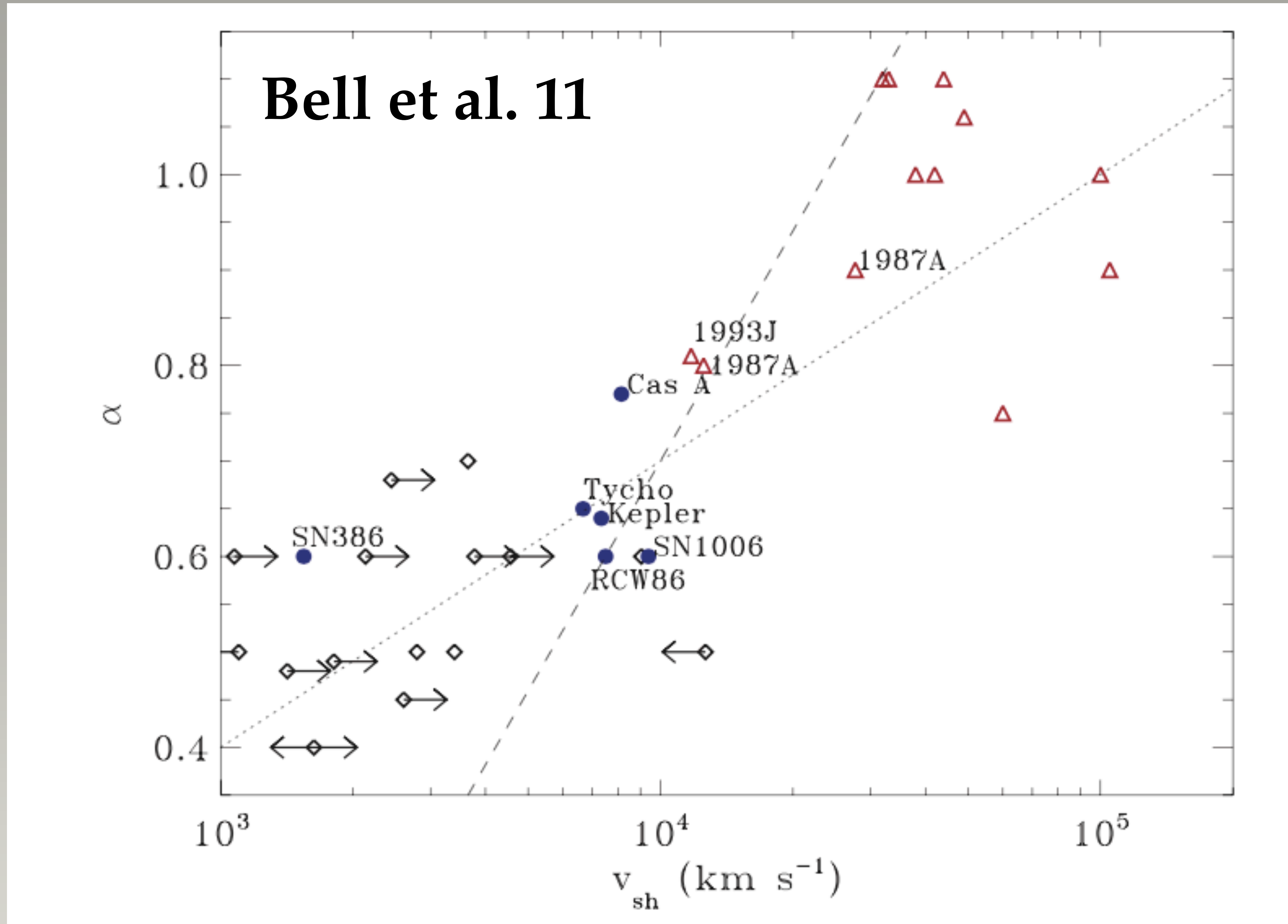




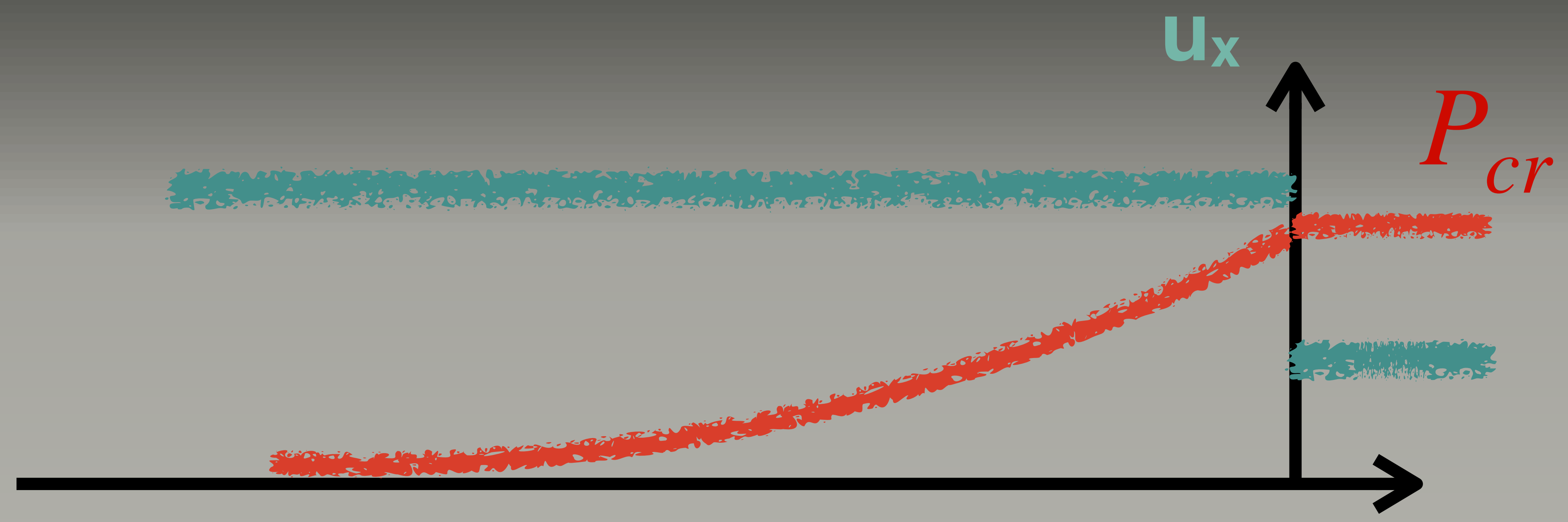
Galactic (◇) Historic (●) Extragalictic (△)

But SNR show a range of values…..

How universal is the DSA prediction

1. **Non-linear feedback: (Eichler 1979)**

- **CR pressure gradient does work on incoming gas**
- **Larger total compression, smaller shock jump**
- **Causes concave spectra**



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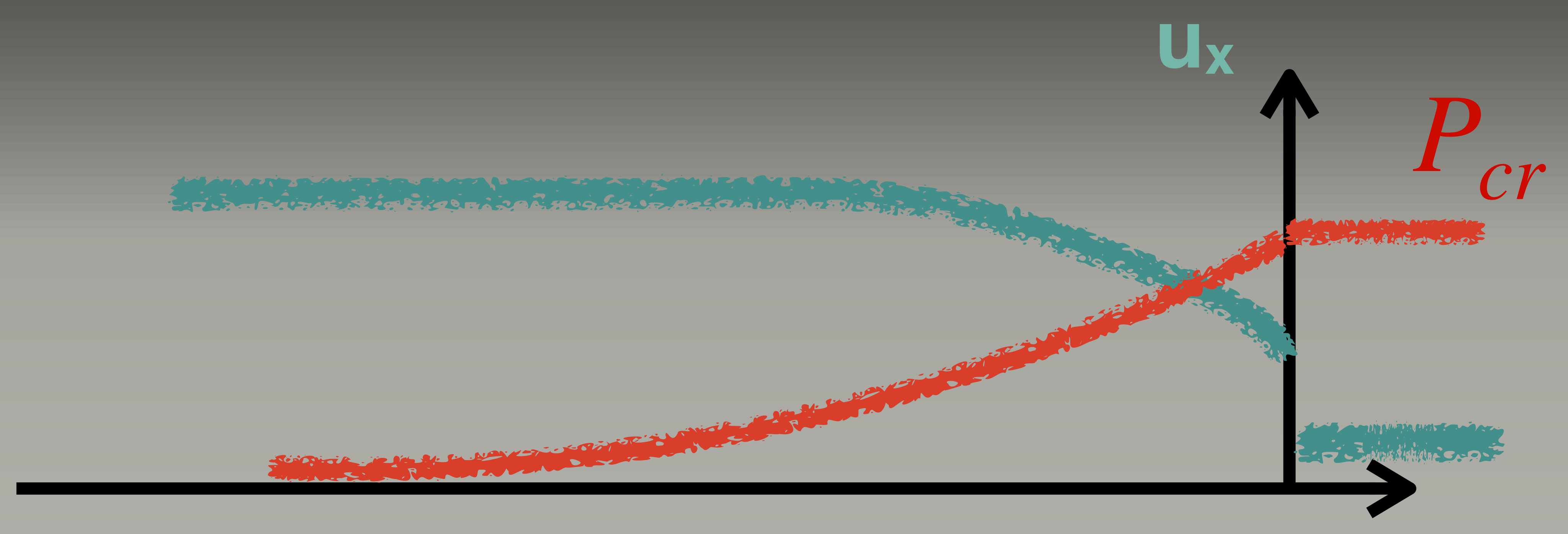

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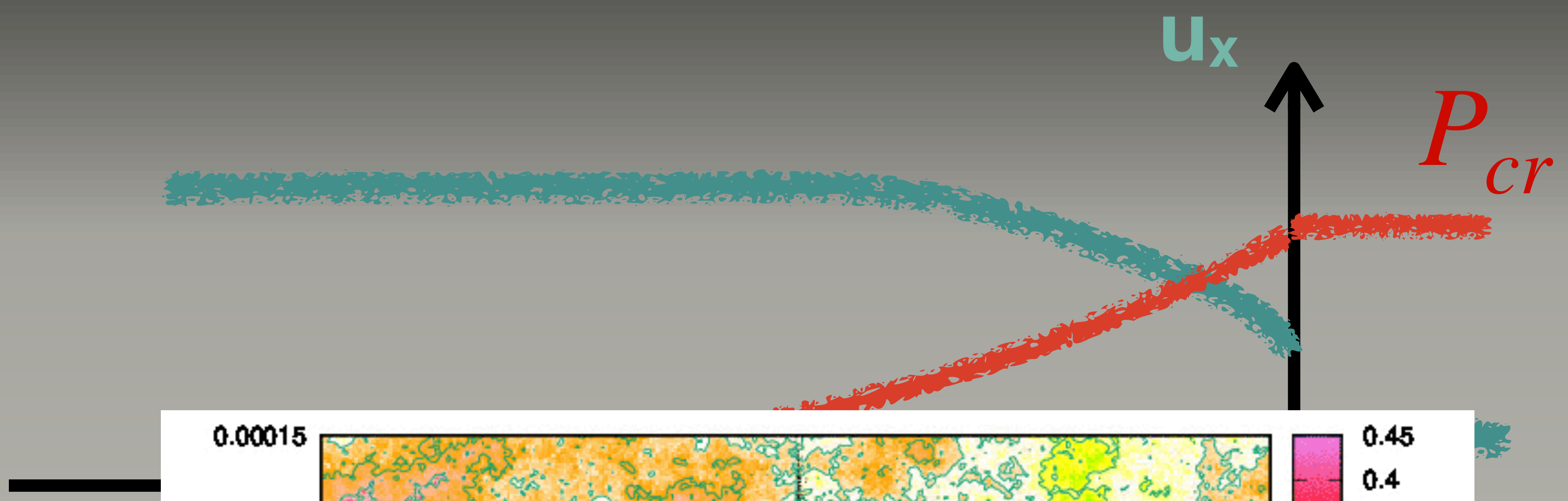

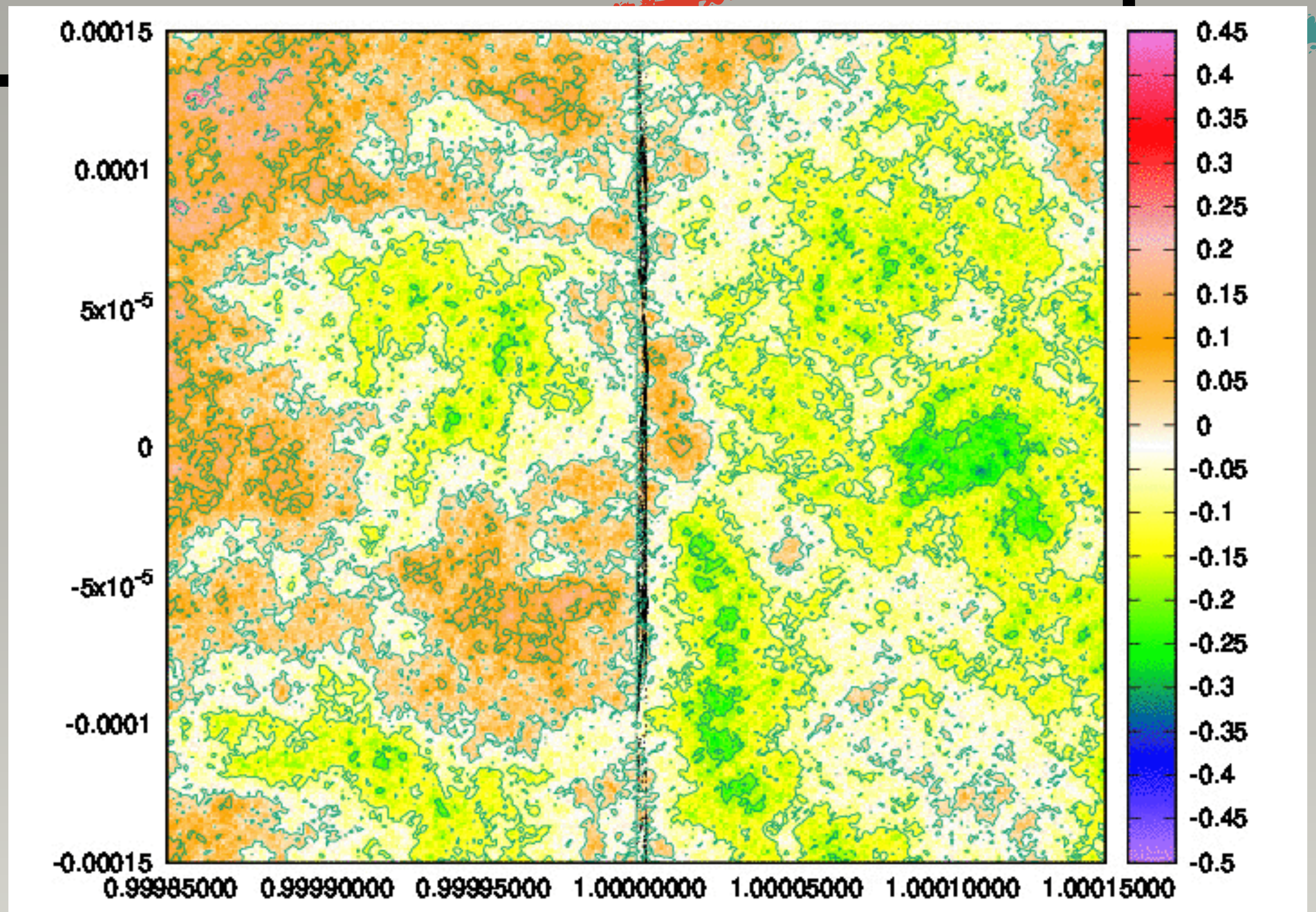

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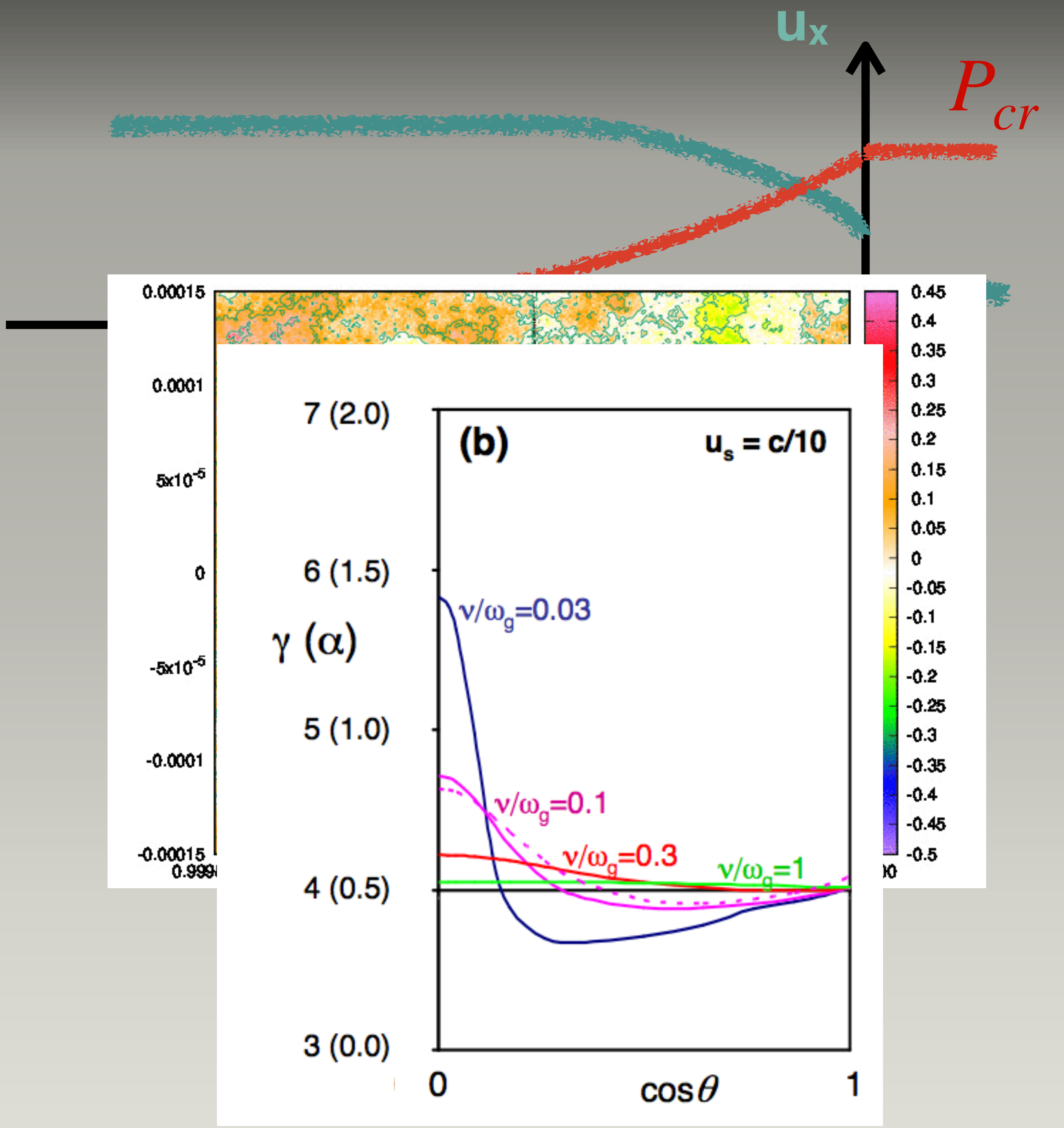
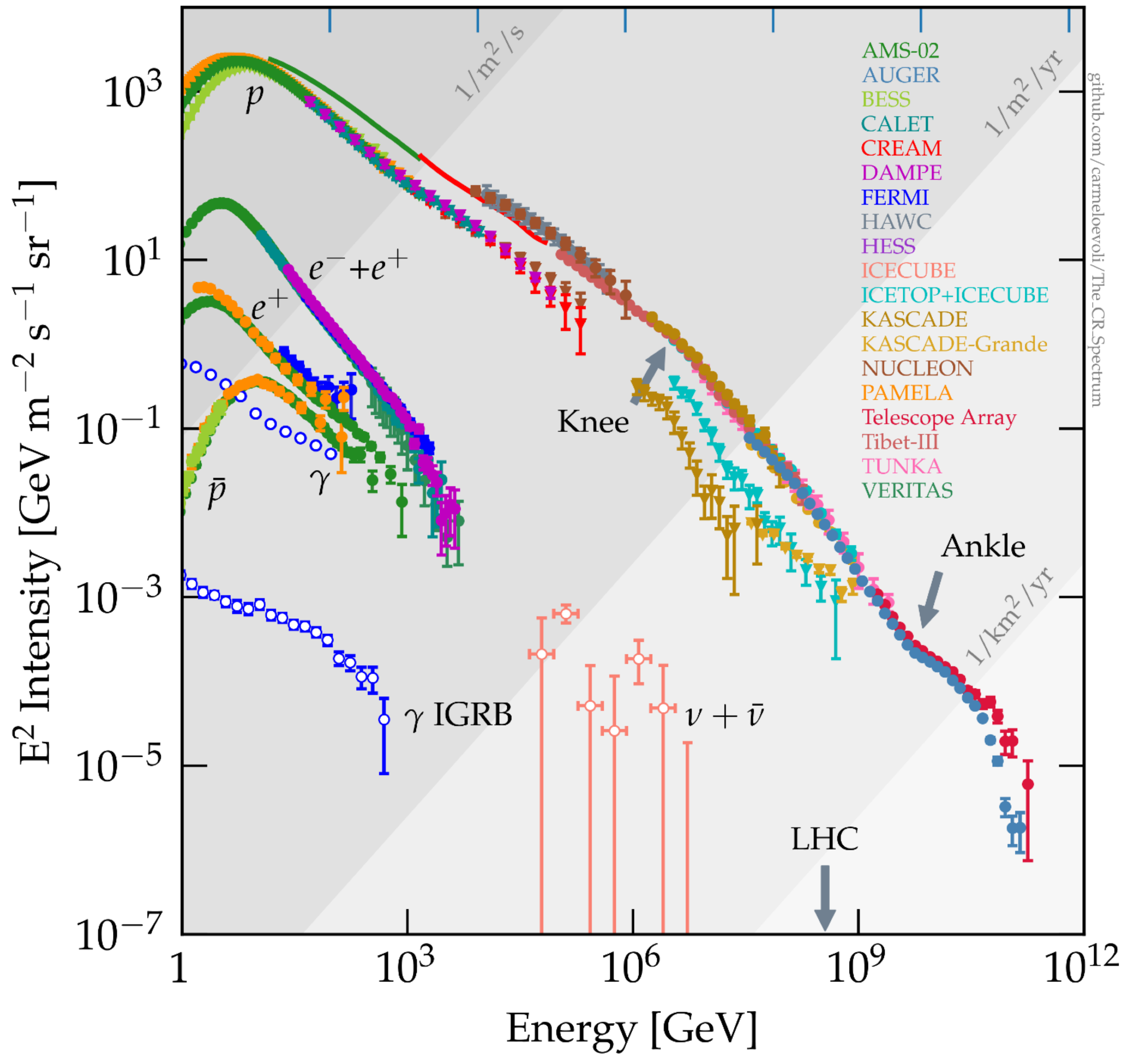
3. **Shock Obliquity (Bell et al. 11)**

- Large scale field introduces additional drifts
- Can harden/soften spectrum



A Galactic CR story (a personal theory)



- Supernova remnants can produce the bulk of Galactic CRs with energy < PeV
- The highest energies are achieved in young fast SNRs in dense environments
- Above the knee requires special sources (Micro-quasars? Massive Stellar Clusters? Something in the Galactic Centre?)
- Energies above the ankle remain a puzzle. We need to consider alternative acceleration processes

- Confirmation requires multi-messenger detections - gamma-rays and neutrinos

Coming up……

- Alternative acceleration methods
- Relativistic shocks
- Emission processes & cooling
- Non-thermal emission spectra

Appendices

A digression into Fokker-Planck theory

Following a collision between scatterers moving relative to one another $\frac{\Delta p}{p} \approx \frac{u_{rel}}{v} \Delta\mu \ll 1$

For Fermi's clouds, $u_{rel} = \pm\, u_{cloud}$ **and** $\Delta\mu = \pm\, 2$.
Crucially, the fractional change in the <u>magnitude</u> of momentum is less than the change in angle
<u>PARTICLES ISOTROPISE FASTER THAN THEY CHANGE ENERGY</u>

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The small changes in angle/momentum naturally lend themselves to a Fokker-Planck treatment.
Consider the particle phase space density $dN = f(\mathbf{x}, \mathbf{p}, t)\ d^3x\ d^3p$**. We consider**

$$f(\boldsymbol{x}, \boldsymbol{p}, t) = \int f(\boldsymbol{x} - \mathbf{v}\Delta t, \boldsymbol{p} - \boldsymbol{\Delta p}, t - \Delta t)\ W(\boldsymbol{p} - \boldsymbol{\Delta p}, \boldsymbol{\Delta p})\ d^3(\Delta p)$$

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$W(\boldsymbol{p}, \boldsymbol{\Delta p})$ **is the probability* that a particle changes its momentum from** $\boldsymbol{p} \to \boldsymbol{p} + \boldsymbol{\Delta p}$ **in a time** Δt

***appropriately normalised:** $\int W(\boldsymbol{p}, \boldsymbol{\Delta p})\ d^3(\Delta p) = 1$

A digression into Fokker-Planck theory

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Rearranging…. $$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f = -\frac{\partial}{\partial p_i}\left\{ f(\boldsymbol{p},t)\left\langle \frac{\Delta p_i}{\Delta t}\right\rangle\right\} + \frac{1}{2}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\left\{ f(\boldsymbol{p},t)\left\langle \frac{\Delta p_i \Delta p_j}{\Delta t}\right\rangle\right\}$$

Where $$\left\langle \frac{\Delta p_i}{\Delta t}\right\rangle = \frac{1}{\Delta t}\int \boldsymbol{\Delta p} W(\boldsymbol{p},\boldsymbol{\Delta p})\; d^3(\Delta p)$$ and $$\left\langle \frac{\Delta p_i \Delta p_j}{\Delta t}\right\rangle = \frac{1}{\Delta t}\int \Delta p_i \Delta p_j W(\boldsymbol{p},\boldsymbol{\Delta p})\; d^3(\Delta p)$$

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In astrophysical plasmas we assume the scatterer is heavier than the ``scatteree". In such cases it makes sense that $W(\boldsymbol{p} - \boldsymbol{\Delta p}, \boldsymbol{\Delta p}) = W(\boldsymbol{p}, -\boldsymbol{\Delta p})$ **(detailed balance)**

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Or on integration: $1 = 1 - \Delta t \underbrace{\frac{\partial}{\partial p_i}\left\{ \left\langle \frac{\Delta p_i}{\Delta t} \right\rangle - \frac{1}{2}\frac{\partial}{\partial p_j}\left\langle \frac{\Delta p_i \Delta p_j}{\Delta t} \right\rangle \right\}}_{=\text{const}=0}$ i.e. friction and diffusion are related

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Finally we have: $\frac{df}{dt} = \frac{\partial}{\partial p_i}\left(D_{ij}\frac{\partial f}{\partial p_j}\right)$ **where** $D_{ij} = \frac{1}{2}\left\langle \frac{\Delta p_i \Delta p_j}{\Delta t} \right\rangle = \frac{1}{2\Delta t}\int \Delta p_i \Delta p_j W(\boldsymbol{p}, \boldsymbol{\Delta p})\, d^3(\Delta p)$

From angular scattering to spatial diffusion

Move to spherical coordinates $(p_x, p_y, p_z) \to (p, \mu, \phi)$, where $\mu = \cos\theta$ is the particle pitch angle relative to the local magnetic field

$$\frac{df}{dt} = \frac{\partial}{\partial p_i}\left(D_{ij}\frac{\partial f}{\partial p_j}\right) = \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 D_{pp}\frac{\partial f}{\partial p}\right) + \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f}{\partial \mu}\right) + ..$$

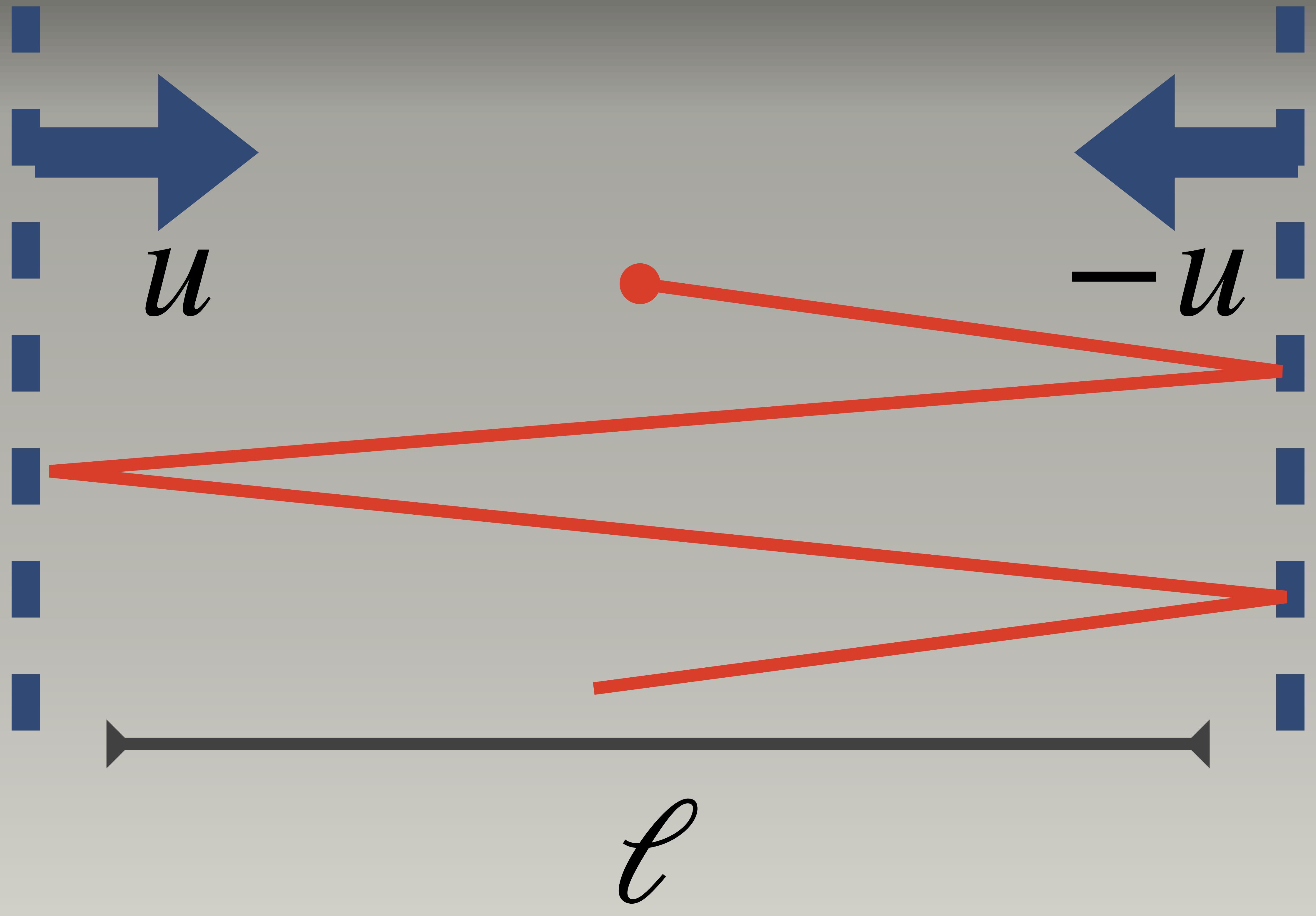
Focus on $D_{\mu\mu}$. On physical grounds, $D_{\mu\mu}$ should vanish at $\mu = \pm 1$. We consider $D_{\mu\mu} = \frac{\nu}{2}(1-\mu^2)$.

Noting the Legendre polynomials are the eigenfunctions of the operator $\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial f}{\partial x}\right)$ suggests looking for solutions that look like $f(p,\mu) = f_0(p) + \mu f_1(p) + (1-\mu^2) f_2(p) + \ldots$

It follows that $\quad \frac{\partial f_0}{\partial t} + \frac{\mathrm{v}}{3}\frac{\partial f_1}{\partial x} = 0 \qquad \frac{\partial f_1}{\partial t} + \mathrm{v}\frac{\partial f_0}{\partial x} = -\nu f_1$ **with steady solution** $\quad f_1 = -\frac{\mathrm{v}}{\nu}\frac{\partial f_0}{\partial x}$

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial x}\left(D_{xx}\frac{\partial f_0}{\partial x}\right)$$ **where** $\quad D_{xx} = \frac{\mathrm{v}^2}{3\nu}$

Simple derivation of adiabatic gains/losses



Consider a FAST particle bouncing off SLOW converging parallel walls. Recall:

$$\frac{\Delta p}{p} = -2\frac{\mathbf{v}\cdot\mathbf{U}_{\rm cloud}}{v^2} = 2(u/v)\ \cos\theta$$

Time between collisions $\Delta t = \ell/v\cos\theta$

$$\frac{\Delta p}{\Delta t} = 2\frac{up}{\ell}\ \cos^2\theta$$

$$\left\langle\frac{\Delta p}{\Delta t}\right\rangle = \frac{\int d\Omega\, f\ 2(up/\ell)\ \cos^2\theta}{\int d\Omega\, f} = \frac{2}{3}\frac{up}{\ell}$$

But $\nabla\cdot\mathbf{u} = -2u/\ell$ $\qquad \left\langle\frac{\Delta p}{\Delta t}\right\rangle = -\frac{p}{3}\nabla\cdot\mathbf{u}$

Putting it together

We make 2 key assumptions.

A) Particles are approximately isotropic

B) They are strongly coupled to the fluid

It follows that the evolution of the particles with v $\gg u$ is well described by the transport equation for the isotropic component of the particle distribution:

$$\underbrace{\frac{\partial f_0}{\partial t} + \mathbf{u} \cdot \nabla f_0}_{\text{Advection}} = \underbrace{\frac{\partial}{\partial x_i}\left(D_{ij} \frac{\partial f_0}{\partial x_j} \right)}_{\text{Diffusion}} + \underbrace{\frac{1}{3} \nabla \cdot \mathbf{u}\; p \frac{\partial f_0}{\partial p}}_{\text{Adiabatic loss/gain}} + \underbrace{\frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial f}{\partial p} \right)}_{\text{Stochastic acceleration}} + \underbrace{\textbf{sources/sinks}}_{\text{Radiative losses / injection}}$$

Note, D_{ij} is the diffusion tensor, which can take different values relative to the mean local magnetic field (recall diffusion along the magnetic field is a lot easier than diffusion across it).